Mathematik

Rigidity properties in coarse geometry

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1 Introduction

Coarse geometry is the central subject of this thesis. This is the study of coarse spaces, i.e. spaces with a large-scale geometric structure. For example, we can turn a metric space into a coarse space by forgetting the fine-scale geometry. The main aim of this thesis is to develop *rigidity* in the setting of coarse geometry.

This thesis contains a wide range of original results. We prove combinatorial results about graph sequences with geometric property (T). We give an in-depth study of geometric property (T) for non-discrete spaces. We develop property (T) for bornological groups and give various applications. We make an important starting point for the study of amenability for bornological groups. We also give new criteria for when a sequence of graphs can be embedded into a Hilbert space.

We will now give a more detailed overview of the chapters of this thesis. In Chapter 2 we recall the background required for the results in the subsequent chapters. That chapter contains no new results.

Chapters 3 and 4 are concerned with *geometric property (T)*. This was introduced in [57] as a property for *uniformly locally finite* coarse spaces. In Chapter 3, which is based on my paper [55], we consider combinatorial properties that are shared by all sequences of graphs with geometric property (T). In particular, we show that such a sequence of graphs always contains a large amount of small cycles. Furthermore, we show that every expander sequence which is a small perturbation of a sequence with geometric property (T) also has geometric property (T).

In Chapter 4, which is based on my paper [54], we develop geometric property (T) for non-discrete spaces, i.e. spaces which need not be locally finite. It turns out that the definition of geometric property (T) for this class of spaces is surprisingly technical, as we first need to construct a measure on the space with desirable properties, and we also need to show that the definition is independent of this measure. We prove various results about geometric property (T) in this setting. One particularly important result is invariance under coarse equivalence. The theory developed in this chapter allows geometric property (T) to be studied much more widely than before. In particular we consider geometric property (T) for (sequences of) Riemannian manifolds and for warped cones.

In Chapter 5, which is based on the joint preprint [51] with Romain Tessera, we develop Kazhdan's property (T) in the context of coarse geometry. We define property (T) for *bornological groups*, i.e. groups with a left-invariant coarse structure. We show that this generalises the classical definition of property (T) for locally compact groups. We derive numerous consequences, such as the fact that property (T) is open in the marked group topology, and that property (T) holds for any compactly generated locally compact group where every affine action on a Hilbert space has almost invariant vectors. These results continue to hold in the more general setting of actions on uniformly convex Banach spaces. We also prove a version of Serre's theorem about the stability of property (T) under central extensions. We introduce the bounded product of a sequence of groups and investigate when it has property (T) as a bornological group and as a topological group. Many results in this chapter depend on finding a suitable notion of the *centre* of a bounded set in a (Hilbert) space.

In Chapter 6 we use a similar technique to prove a result about amenable groups acting on measure spaces. We show that under a mild condition, every measure-preserving action of an amenable locally compact group on a finitely-additive measure space admits an invariant mean. This is significant because it gives a characterisation of amenability that does not depend on the Haar measure of the group. In particular this result could potentially be used to introduce a notion of amenability for bornological groups. This could be a fruitful direction for future work.

The last two chapters of this thesis contain partial results on an unfinished, challenging project. The aim of this project was to find a sequence of graphs with uniformly bounded degree which allows a coarse embedding into some ℓ^p-space but not a Hilbert space. This is a fundamental open question in the field (see e.g. [2]). As shown in [50], such a sequence necessarily contains a *generalised expander*. In Chapter 7 we study these generalised expanders in detail. We introduce *ρ-expanders* and we show that their covers cannot be coarsely embedded into a Hilbert space. It may be possible to generate ρ-expanders whose covers can be coarsely embedded into an ℓ^p-space using random graph methods. This would require a model of a random graph embedded in a higher-dimensional torus.

In Chapter 8 we develop one possible method to generate random graphs embedded in a manifold. Since it is difficult to analyse the behaviour of random graphs embedded in a torus, the problem of finding sequences of graphs with uniformly bounded degree which embed into an ℓ^p-space but not into a Hilbert space remains open. However, Chapter 8 contains results about random graphs embedded in $\mathbb{R}^m$ which are of independent interest.

Chapter 2 contains no new results. Chapter 5 is based on the preprint [51], which is joint work with Romain Tessera. Large parts of this paper are reproduced verbatim. All other results in this thesis are my own. Chapter 3 is based on the paper [55] and large parts of it are reproduced verbatim. Chapter 4 is based on the paper [54] and large parts of it are reproduced verbatim. The results in Chapters 6, 7 and 8 appear for the first time in this thesis.

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2 Preliminaries

In this chapter we introduce the background that is used in the subsequent chapters of this thesis. This chapter contains no new results.

2.1 Coarse geometry

2.1.1 Coarse spaces

Coarse geometry is a central topic in this thesis. It is the study of coarse spaces. These are spaces equipped with a "large-scale" geometric structure. They were introduced in [23]. We take most results in this section from [44]. We start with the definition and then we will give some motivation and examples. In what follows, we use the following notation: for any set X and a subset $E \subseteq X \times X$ we denote

$$E^{-1} = \{(x, y) \mid (y, x) \in E\}.$$

For subsets $E, F \subseteq X \times X$ we denote

$$E \circ F = \{(x, z) \in X \times X \mid \text{there is } y \in X \text{ such that } (x, y) \in E \text{ and } (y, z) \in F\}.$$

Definition 2.1.1. Let X be a set. A *coarse structure* on X is a subset $\mathcal{E} \subseteq \mathcal{P}(X \times X)$ satisfying the following conditions:

(i) the diagonal $\{(x, x) \mid x \in X\}$ is an element of $\mathcal{E}$;

(ii) for $F \in \mathcal{E}$ and $E \subseteq F$ we have $E \in \mathcal{E}$;

(iii) for $E \in \mathcal{E}$ we have $E^{-1} \in \mathcal{E}$;

(iv) for $E, F \in \mathcal{E}$ we have $E \cup F \in \mathcal{E}$;

(v) for $E, F \in \mathcal{E}$ we have $E \circ F \in \mathcal{E}$.

A set equipped with a coarse structure is called a *coarse space*. Instead of writing $E \in \mathcal{E}$, we may write "E is a controlled set" if the coarse structure $\mathcal{E}$ is clear from the context.

Note that $\circ$ is an associative operation: for $E, F, G \subseteq X \times X$ we have $(E \circ F) \circ G = E \circ (F \circ G) = \{(x, w) \mid \text{there are } y, z \in X \text{ such that } (x, y) \in E,\ (y, z) \in F,\ (z, w) \in G\}$. Thus we may write this simply as $E \circ F \circ G$.

Example 2.1.2. Every set X can be equipped with a trivial coarse structure which consists of all subsets of the diagonal. We can also equip X with the coarse structure containing all subsets of $X \times X$.

Many important examples come from metric spaces. Consider a set X with a metric $d\colon X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. Note that for us, a metric is allowed to take infinity as a value. This induces a coarse structure $\mathcal{E}_d$, given by

$$\mathcal{E}_d = \{E \subseteq \mathcal{P}(X \times X) \mid \sup\{d(x, y) \mid (x, y) \in E\} < \infty\}.$$

It is easy to see that this is indeed a coarse structure; for example, to check condition (v) from the definition, suppose that E and F are controlled. Then for all $(a,c) \in E \circ F$ there is $b \in X$ with $(a,b) \in E$ and $(b,c) \in F$, thus $d(a,c) \leq d(a,b) + d(b,c) \leq \sup\{d(x,y) \mid (x,y) \in E\} + \sup\{d(x,y) \mid (x,y) \in F\}$. Thus $\sup\{d(x,y) \mid (x,y) \in E \circ F\} \leq \sup\{d(x,y) \mid (x,y) \in E\} + \sup\{d(x,y) \mid (x,y) \in F\} < \infty$, and therefore $E \circ F$ is also controlled.

We can think of controlled sets as sets whose elements are within a bounded distance from the diagonal.

For any metric d, we can define a new metric $\tilde{d}$ by

$$\tilde{d}(x,y) = \begin{cases} 0 & \text{if } x = y; \\ \max(1, d(x,y)) & \text{else.} \end{cases}$$

Then for any subset $E \subseteq X \times X$ that is not contained in the diagonal, we see that

$$\sup\{\tilde{d}(x,y) \mid (x,y) \in E\} = \max(1, \sup\{d(x,y) \mid (x,y) \in E\}).$$

So we see that the coarse structures $\mathcal{E}_d$ and $\mathcal{E}_{\tilde{d}}$ coincide.

The metric $\tilde{d}$ forgets all 'small-scale' structure of the space; indeed, different points are always at distance at least 1 now, so there can be no notion of convergence, for example. On the other hand, the large-scale structure of the space is maintained. We still have a notion of bounded sets.

Definition 2.1.3. Let $(X, \mathcal{E})$ be a coarse space. A subset $C \subseteq X$ is called *bounded* if $C \times C$ is a controlled set.

Now it is easy to see that for a metric space (X, d), a subset C is bounded for $\mathcal{E}_d$ exactly when it has a bounded diameter.

The intersection of a family of coarse structures is again a coarse structure. Thus we can make the following definition.

Definition 2.1.4. Let X be a set and let $\mathcal{E}' \subseteq \mathcal{P}(X \times X)$. The coarse structure *generated* by $\mathcal{E}'$ is the smallest coarse structure containing $\mathcal{E}'$. It equals the intersection of all coarse structures containing $\mathcal{E}'$.

A coarse structure $\mathcal{E}$ is *countably generated* if there is a countable subset of $\mathcal{P}(X \times X)$ that generates it.

Many, but not all coarse spaces come from metric spaces. The following result and proof have been taken from [44, Theorem 2.55].

Proposition 2.1.5. *A coarse space $(X, \mathcal{E})$ is countably generated if and only if there is a metric d on X (possibly taking infinity as a value) such that $\mathcal{E} = \mathcal{E}_d$.*

Proof. Suppose $\mathcal{E} = \mathcal{E}_d$. For $r \in \mathbb{N}$ let E_n be the controlled set $\{(x,y) \in X \times X \mid d(x,y) \leq n\}$. Then the E_n generate the coarse structure $\mathcal{E}$ since each controlled set is contained in one of the E_n.

Conversely, suppose that $E_1, E_2, \ldots$ generate $\mathcal{E}$. Define recursively F_0 to be the diagonal and

$$F_n = (F_{n-1} \circ F_{n-1}) \cup E_n \cup E_n^{-1}$$

for $n \geq 1$. Note that all F_n are symmetric and controlled, they contain the diagonal and $F_{n-1} \subseteq F_n$.

Now let $d(x,y) = \inf\{n \mid (x,y) \in F_n\}$. This is infinite if $(x,y) \notin \bigcup_n F_n$ and 0 if and only if $x = y$. Since the F_n are symmetric, d is symmetric. We show that d satisfies the triangle inequality.

Let $x, y, z \in X$ be distinct elements and suppose $(x, y) \in F_k$ and $(y, z) \in F_l$. Then $(x, z) \in F_k \circ F_l \subseteq F_{\max(k,l)} \circ F_{\max(k,l)} \subseteq F_{\max(k,l)+1}$. Thus $d(x, z) \leq \max(d(x, y), d(y, z)) + 1 \leq d(x, y) + d(y, z)$. This shows that d is a metric.

Now we show that $\mathcal{E} = \mathcal{E}_d$. First note that all F_n are in $\mathcal{E}$ by construction. Any $E \in \mathcal{E}_d$ is a subset of F_n where $n = \sup\{d(x, y) \mid (x, y) \in E\}$, so $\mathcal{E}_d \subseteq \mathcal{E}$. Moreover, all E_n are contained in $\mathcal{E}_d$, since $\sup\{d(x, y) \mid (x, y) \in E_n\} \leq n$. So $\mathcal{E}_d$ is a coarse structure containing all E_n, showing that $\mathcal{E}_d \supseteq \mathcal{E}$. □

The primary reason to consider coarse spaces, rather than metric spaces, is that there is more flexibility in the morphisms between coarse spaces. A morphism between coarse spaces is called a controlled map (in [44] this is called a bornologous map).

Definition 2.1.6. Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces. A function $f\colon X \to Y$ is called *controlled* if for all $E \in \mathcal{E}$ we have

$$f_*(E) = \{(f(x), f(x')) \mid (x, x') \in E\} \in \mathcal{F}.$$

For later reference we show what this concept means for functions between metric spaces.

Proposition 2.1.7. *Let (X, d) and (Y, d') be metric spaces. Equip X and Y with the corresponding coarse structures $\mathcal{E}_d$ and $\mathcal{E}_{d'}$. Then a function $f\colon X \to Y$ is controlled if and only if there is a function $\rho\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that whenever $d(x, x') \leq r < \infty$ we have $d'(f(x), f(x')) \leq \rho(r)$.*

Proof. Suppose f is controlled. For $r \geq 0$ let $E_r = \{(x, x') \in X \times X \mid d(x, x') \leq r\}$. This is a controlled set for X, so $f_*(E_r)$ is also controlled. Let $\rho(r) = \sup\{d'(y, y') \mid (y, y') \in f_*(E_r)\}$. Then for $x, x' \in X$ with $d(x, x') \leq r$ we have $(x, x') \in E_r$ so $(f(x), f(x')) \in f_*(E_r)$, so $d'(f(x), f(x')) \leq \rho(r)$.

Conversely suppose that a suitable ρ exists. Let $E \subseteq X \times X$ be a controlled set. Let $r = \sup\{d(x, x') \mid (x, x') \in E\} < \infty$. Then for all $(x, x') \in E$ we have $d'(f(x), f(x')) \leq \rho(r)$. Thus $\sup\{d'(y, y') \mid (y, y') \in f_*(E)\} \leq \rho(r)$, which shows that $f_*(E)$ is controlled, so f is controlled. □

We see that controlled maps are quite flexible, when compared for example with isometries or with Lipschitz functions.

2.1.2 Coarse equivalences

Next we will consider coarse equivalences.

Definition 2.1.8. Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces. A map $f\colon X \to Y$ is a *coarse equivalence* if f is controlled and there is a controlled map $g\colon Y \to X$ such that

$$\{(g \circ f(x), x) \mid x \in X\} \in \mathcal{E}$$

and

$$\{f \circ g(y), y) \mid y \in Y\} \in \mathcal{F}.$$

We say that $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are *coarsely equivalent* if such a map exists.

Example 2.1.9. Let $X = \mathbb{Z}^n$ and $Y = \mathbb{R}^n$, with the coarse structures induced from the Euclidean metric. Then the embedding $f\colon \mathbb{Z}^n \to \mathbb{R}^n$ is a coarse equivalence. We may take g to be the function $g(x_1, \ldots, x_n) = (\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor)$.

While coarse equivalences are quite flexible, there are still many interesting properties that are invariant under coarse equivalences. We will see examples of this later in Section 2.4.

A subset Y of a coarse space $(X, \mathcal{E})$ has an induced coarse structure $\{E \cap (Y \times Y) \mid E \in \mathcal{E}\}$. An easy way to create coarsely equivalent spaces is to take a *coarsely dense* subset, equipped with this induced coarse structure.

Definition 2.1.10. Let $(X, \mathcal{E})$ be a coarse space and let $Y \subseteq X$. We say Y is *coarsely dense* if there is a controlled set $E \in \mathcal{E}$ such that for all $x \in X$ there is $y \in Y$ with $(x, y) \in E$.

Lemma 2.1.11. *Let $(X, \mathcal{E})$ be a coarse space and $Y \subseteq X$ a coarsely dense subspace. Equip Y with the induced coarse structure $\mathcal{F}$. Then the embedding $i \colon Y \hookrightarrow X$ is a coarse equivalence.*

Proof. It is clear that i is a controlled map. Let $E \in \mathcal{E}$ such that for all $x \in X$ there is $y \in Y$ with $(x, y) \in E$. For all $x \in X$ we choose $g(x) \in Y$ with $(x, g(x)) \in E$.

We show that g is controlled. Let $F \in \mathcal{E}$ and $(x, x') \in F$. Then $(g(x), x) \in E^{-1}$, $(x, x') \in F$ and $(x', g(x')) \in E$. Therefore $(g(x), g(x')) \in E^{-1} \circ F \circ E$. Thus we have $g_*(F) \subseteq E^{-1} \circ F \circ E$, showing that g is controlled.

Now $\{(x, i(g(x)) \mid x \in X\} \subseteq E$ is controlled, and $\{y, g(i(y)) \mid y \in Y\} \subseteq E \cap (Y \times Y)$ is controlled, so i is a coarse equivalence. $\square$

Moreover, every coarse equivalence can be built up out of two coarse equivalences of this form and a coarse isomorphism (i.e. a bijection between coarse spaces that preserves the coarse structure). This is convenient because coarsely dense subspaces are easier to understand than the general case of coarse equivalence.

Proposition 2.1.12 ([58], Lemma 4.1)**.** *Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces and $f \colon X \to Y$ be a coarse equivalence. Then there are a coarsely dense $X' \subseteq X$, a coarsely dense $Y' \subseteq Y$, with induced coarse structures $\mathcal{E}'$ and $\mathcal{F}'$ respectively, and a bijection $h \colon X' \to Y'$ satisfying $h_*(\mathcal{E}') = \mathcal{F}'$.*

Proof. Let g be as in Definition 2.1.8. Let X' be the image of g. This is coarsely dense in X because the set $\{(x, g(f(x)) \mid x \in X\}$ is controlled. For each $x \in X'$ choose $h(x) \in Y$ with $g(h(x)) = x$. This defines an injection from X' to Y. Let Y' be its image. The set $F_1 = \{(y, f(g(y)) \mid y \in Y\}$ is controlled and so is its subset $F_2 = \{(h(g(y)), f(g(h(g(y))))) \mid y \in Y\}$. We have $f(g(h(g(y)))) = f(g(y))$. So the set $\{(y, h(g(y))) \mid y \in Y\}$ is contained in the composition $F_1 \circ F_2^{-1}$ and is controlled. So Y' is coarsely dense in Y.

Now we need to show that h and its inverse are controlled. We have $\{(h(x), f(x)) \mid x \in X'\} = \{(h(g(y)), f(g(y)) \mid y \in Y\} = F_2$ which is controlled. Then for any $E \in \mathcal{E}'$ we have $h_*(E) \subseteq F_2 \circ f_*(E) \circ F_2^{-1}$, showing that h is controlled.

The inverse of h is also controlled, since it is a restriction of g. So h is a coarse isomorphism between $(X', \mathcal{E}')$ and $(Y', \mathcal{F}')$. $\square$

2.1.3 Bornological groups

To study groups from the perspective of coarse geometry, it is suitable to define a *bornology* on the group.

Definition 2.1.13. Let G be a group. A *bornology* is a subset $\mathcal{B} \subseteq P(G)$ satisfying the following conditions:

(i) For each $g \in G$ we have $\{g\} \in \mathcal{B}$.

(ii) For $A \in \mathcal{B}$ and $B \subseteq A$ we have $B \in \mathcal{B}$.

(iii) For $A \in \mathcal{B}$ we have $A^{-1} = \{a^{-1} \mid a \in A\} \in \mathcal{B}$.

(iv) For $A, B \in \mathcal{B}$ we have $A \cup B \in \mathcal{B}$.

(v) For $A, B \in \mathcal{B}$ we have $A \cdot B = \{ab \mid a \in A, b \in B\} \in \mathcal{B}$.

Instead of writing $A \in \mathcal{B}$ we may write A *is bounded* if the bornology is clear from the context. A *bornological group* is a group equipped with a bornology.

Example 2.1.14. For any group G we can define the *discrete bornology* which consists of all finite subsets of G, and the trivial bornology $\mathcal{B} = \mathcal{P}(G)$ for which every subset of G is bounded.

Bornologies are closely related to coarse structures. Any bornological group $(G, \mathcal{B})$ has a left-invariant coarse structure $\mathcal{E}_l$, for which a subset $E \subseteq G \times G$ is controlled if and only if the set $\{x^{-1}y \mid (x, y) \in E\}$ is bounded. It is generated by the sets $E_A = \{(ga, g) \mid a \in A, g \in G\}$ for bounded $A \subseteq G$. Similarly, there is a right coarse structure generated by the sets $\{(ag, g) \mid a \in A, g \in G\}$. Since they may not coincide in general, the coarse structure is not the most convenient and we prefer to work with the bornology directly. Many concepts of coarse geometry can be directly stated in terms of the bornology.

Definition 2.1.15. Let $(G, \mathcal{B})$ be a bornological group and let $H \subseteq G$ be a subgroup. We say that H is *coarsely dense* if there is a bounded set A satisfying $HA = G$.

It can easily be checked that this coincides with our earlier definition of coarsely dense when G is equipped with either the left- or right invariant coarse structure induced by $\mathcal{B}$.

Topological groups are an important source of examples for bornological groups.

Example 2.1.16. Let (G, τ) be a (Hausdorff) topological group. The *pre-compact bornology* is the bornology $\mathcal{K}$ consisting of all subsets of G whose closure is compact.

In Section 5.1.1 we will see another way to construct a bornology on a topological group.

2.2 Property (T)

Much of this thesis will be about variations of *property (T)*. We start by introducing the classical setting of property (T), which applies to groups. It concerns the representations of groups on Hilbert spaces. For a Hilbert space $\mathcal{H}$, we denote by $U(\mathcal{H})$ the group of unitary maps:

$$U(\mathcal{H}) = \{T \colon \mathcal{H} \to \mathcal{H} \mid T \text{ continuous and linear, } T^*T = TT^* = I\}.$$

Definition 2.2.1. Let Γ be a group. A *representation* of Γ consists of a Hilbert space $\mathcal{H}$ and a homomorphism $\pi \colon \Gamma \to U(\mathcal{H})$.

Note that the representations we consider are always unitary. If $(\pi, \mathcal{H})$ is a representation of Γ and $g \in \Gamma, v \in \mathcal{H}$, we will sometimes write $g \cdot v$ as an abbreviation for $\pi(g)(v)$ if the map π is clear from the context. Note that $1 \cdot v = v$ and $(g \cdot h) \cdot v = g \cdot (h \cdot v)$, because π is a homomorphism. An important example is the left-regular representation.

Definition 2.2.2. Let Γ be a group. The Hilbert space $l^2\Gamma$ is the set of functions

$$l^2\Gamma = \left\{ f\colon \Gamma \to \mathbb{C} \,\middle|\, \sum_{g\in\Gamma} |f(g)|^2 < \infty \right\},$$

equipped with the inner product

$$\langle f, f' \rangle = \sum_{g\in\Gamma} \overline{f(g)} f'(g).$$

The left-regular representation of Γ is the representation $\lambda\colon \Gamma \to U(l^2\Gamma)$ given by $(\lambda(g)(f))(g') = f(g^{-1}g')$.

Definition 2.2.3. Let Γ be a group and $(\pi, \mathcal{H})$ a representation of Γ. A unit vector $v \in \mathcal{H}$ is called an *invariant vector* if $g \cdot v = v$ for all $g \in \Gamma$.

We can always decompose a representation $(\pi, \mathcal{H})$ into a part where all unit vectors are invariant and a part without invariant vectors: consider the closed subspace $\mathcal{H}_c = \{v \in \mathcal{H} \mid g\cdot v = v \text{ for all } g \in \Gamma\}$. Then for all $g \in \Gamma$ we have $g \cdot \mathcal{H}_c = \mathcal{H}_c$ and also $g \cdot H_c^\perp = \mathcal{H}_c^\perp$. We can then define the representations $\pi_c\colon \Gamma \to U(\mathcal{H}_c)$ by $\pi_c(g)v = v$ and $\pi_c^\perp\colon \Gamma \to U(\mathcal{H}_c^\perp)$ by $\pi_c^\perp(g)v = \pi(g)v$. Then all unit vectors of $\mathcal{H}_c$ are invariant vectors while $(\pi_c^\perp, \mathcal{H}_c^\perp)$ does not have any invariant vectors. Moreover, we have $\pi = \pi_c \oplus \pi_c^\perp$.

Now we would like to have a good notion of what it means for a representation $(\pi, \mathcal{H})$ to have *almost* invariant vectors. We have the following Proposition, which is taken from [5, Proposition 1.1.5].

Proposition 2.2.4. *Let Γ be a group and $(\pi, \mathcal{H})$ a representation of Γ. Let $0 < \varepsilon < \sqrt{2}$ and let $v \in \mathcal{H}$ be a unit vector such that $\|g \cdot v - v\| \leq \varepsilon$ for all $g \in \Gamma$. Then there is an invariant vector $w \in \mathcal{H}$ with $\|v - w\| < \varepsilon$.*

Proof. Let C be the closed convex hull of $\pi(\Gamma)v$. Since C is closed and convex there is a unique vector $u \in C$ with minimal norm: to see this, let u_n be a sequence of vectors with $\lim_n \|u_n\| = \inf_{c\in C}\|c\|$. For any m and n, the midpoint $\frac{u_m+u_n}{2}$ is also in C, therefore $\left\|\frac{u_m+u_n}{2}\right\| \geq \inf_{c\in C}\|c\|$. Using the parallellogram identity we have $\left\|\frac{u_m+u_n}{2}\right\|^2 = \frac{1}{2}\|u_m\|^2 + \frac{1}{2}\|u_n\|^2 - \frac{1}{4}\|u_m - u_n\|^2$. So we have $\frac{1}{4}\|u_m - u_n\|^2 \leq \frac{1}{2}\|u_m\|^2 + \frac{1}{2}\|u_n\|^2 - \inf_{c\in C}\|c\|^2$. This shows that (u_n) is a Cauchy sequence, and since C is closed, it converges to a vector u with $\|u\| = \inf_{c\in C}\|c\|$. This vector is clearly unique, because if there were two different vectors $u, u' \in C$ with minimal norm, then the vector $\frac{u+u'}{2}$ would also be in C and have smaller norm, giving a contradiction.

For any $g \in \Gamma$ we have $\pi(g)\pi(\Gamma)v = \pi(\Gamma)v$, and taking the convex closure of both sides gives $\pi(g)C = C$. Since $\pi(g)$ is a unitary map, and u is the unique vector with minimal norm in C, we see that $\pi(g)u$ is the unique vector with minimal norm in $\pi(g)C = C$. So $\pi(g)u = u$.

Now we have to show that u is non-zero. For all $g \in \Gamma$ we have $\|g \cdot v - v\| \leq \varepsilon$. We have $\|g \cdot v - v\|^2 = 2 - 2\mathrm{Re}(\langle g \cdot v, v\rangle)$, and combining this with the inequality gives

$$\mathrm{Re}(\langle g \cdot v, v\rangle) \geq 1 - \frac{\varepsilon^2}{2}.$$

Since C is the convex closure of $\pi(\Gamma)v$, we get the inequality

$$\mathrm{Re}(\langle w, v\rangle) \geq 1 - \frac{\varepsilon^2}{2}$$

for all $w \in C$. In particular, we have $\mathrm{Re}(\langle u, v\rangle) \geq 1 - \frac{\varepsilon^2}{2} > 0$, so $u \neq 0$. We can then define $x = \frac{u}{\|u\|}$. Then x is an invariant vector. Since $\|u\| \leq 1$ we get

$$\|v - x\|^2 = 2 - 2\mathrm{Re}(\langle x, v\rangle) = 2 - \frac{2\mathrm{Re}(\langle u, v\rangle)}{\|u\|} \leq 2 - \frac{2 - \varepsilon^2}{\|u\|} \leq \varepsilon^2.$$

□

The concept of "almost invariant vectors" only asks the vectors to be almost invariant with respect to the action of group elements in a finite set at the same time.

Definition 2.2.5. Let Γ be a group and $(\pi, \mathcal{H})$ a representation of Γ. We say that the representation has *almost invariant vectors* if for every finite set $F \subseteq \Gamma$ and every $\varepsilon > 0$, there exists a unit vector $v \in \mathcal{H}$ satisfying $\|\pi(g)v - v\| < \varepsilon$ for all $g \in F$.

Now there are groups for which there are representations with almost invariant vectors but without invariant vectors. We can consider the left-regular representation of $\mathbb{Z}$ as an example.

Example 2.2.6. The left-regular representation $\lambda\colon \mathbb{Z} \curvearrowright l^2\mathbb{Z}$ has almost invariant vectors, but no invariant vector. To see this, let $F \subseteq \mathbb{Z}$ be finite and $\varepsilon > 0$. Let n be a large enough integer that $\sqrt{\frac{2|k|}{n}} < \varepsilon$ for all $k \in F$. Let $v_n \in l^2\mathbb{Z}$ be the function given by

$$v_n(m) = \begin{cases} \frac{1}{\sqrt{n}} & \text{if } 1 \leq m \leq n \\ 0 & \text{else.} \end{cases}$$

Then v_n is a unit vector. For $k \in F$ we see that $\lambda(k)v_n - v_n$ is supported on $2|k|$ integers, taking a value of $\pm\frac{1}{\sqrt{n}}$ there, thus $\|\lambda(k)v_n - v_n\| = \sqrt{\frac{2|k|}{n}} < \varepsilon$. This shows that the left-regular representation has almost invariant vectors. There are no invariant vectors, since they would necessarily be a non-zero constant function on $\mathbb{Z}$ and thus not square-summable.

For some groups, however, the existence of almost invariant vectors in a given representation guarantees the existence of an invariant vector.

Definition 2.2.7. A group Γ has *property (T)* if every representation with almost invariant vectors has an invariant vector.

Example 2.2.8. By Proposition 2.2.4, all finite groups have property (T). As we saw in example 2.2.6, $\mathbb{Z}$ does not have property (T). More generally, no infinite amenable groups have property (T); see Proposition 2.3.4. The group $\mathrm{SL}_n(\mathbb{Z})$ of $n \times n$ matrices with integer coefficients and determinant 1 has property (T) if and only if $n \geq 3$ (see [5, Example 1.7.4]).

Property (T) is inherited by quotients.

Proposition 2.2.9. *Let Γ be a group with property (T) and let N be a normal subgroup. Then Γ/N has property (T).*

Proof. Let $\pi\colon \Gamma/N \curvearrowright \mathcal{H}$ be a representation with almost invariant vectors. Precomposing with the projection $\Gamma \twoheadrightarrow \Gamma/N$ gives a representation of Γ with almost invariant vectors. Thus $\mathcal{H}$ has a Γ-invariant vector, which is then also Γ/N-invariant. □

Example 2.2.10. The free group F_n on n generators does not have property (T) for any $n \geq 1$, since there is a surjective homomorphism from F_n to $\mathbb{Z}$.

2.2.1 Property (T) for locally compact groups

So far in this section the group Γ did not have a topology. There is a variant of property (T) that applies to locally compact groups. Here we restrict our attention to continuous representations. Moreover, in the definition of "almost invariant vectors", we consider compact sets instead of finite sets.

Definition 2.2.11. Let (G, τ) be a locally compact group. A *continuous representation* of G is a representation $\pi\colon G \to U(\mathcal{H})$ such that for all $v \in \mathcal{H}$ the map $G \to \mathcal{H},\ g \to \pi(g)v$ is continuous. It has *almost invariant vectors* if for all compact $K \subseteq G$ and all $\varepsilon > 0$ there is a unit vector $v \in \mathcal{H}$ such that $\lVert \pi(g)v - v \rVert < \varepsilon$ for all $g \in K$. We say that (G, τ) has *property (T)* if every continuous representation with almost invariant vectors has an invariant vector.

From Proposition 2.2.4 we know that all compact groups have property (T).

Definition 2.2.12. Let (G, τ) be a locally compact group. A compact set $K \subseteq G$ and an $\varepsilon > 0$ form a *Kazhdan pair* if the following holds for every continuous representation $\pi\colon G \to U(\mathcal{H})$: if there is a unit vector $v \in \mathcal{H}$ satisfying $\lVert \pi(g)v - v \rVert < \varepsilon$ for all $g \in K$, then there is an invariant vector.

Clearly G must have property (T) if a Kazhdan pair exists. It turns out that the converse is true as well.

Proposition 2.2.13. *Let (G, τ) be a locally compact group with property (T). Then there exists a Kazhdan pair (K, ε).*

Proof. Suppose it is not true. Then for each pair (K, ε) consisting of a compact subset of the group and a positive real number, there exists a representation $\pi_{K,\varepsilon}\colon G \curvearrowright U(\mathcal{H}_{K,\varepsilon})$ without invariant vectors and a unit vector $v_{K,\varepsilon} \in \mathcal{H}_{K,\varepsilon}$ such that $\lVert \pi(g)v_{K,\varepsilon} - v_{K,\varepsilon} \rVert < \varepsilon$ for $g \in K$. Now consider the direct sum representation

$$\pi = \bigoplus_{K,\varepsilon} \pi_{K,\varepsilon}.$$

This contains the $v_{K,\varepsilon}$ and thus it has almost invariant vectors. Since G has property (T), there must be an invariant vector v. This must have a non-zero component v_0 in some $\mathcal{H}_{K,\varepsilon}$. But then rescaling v_0 gives an invariant vector in $\mathcal{H}_{K,\varepsilon}$, a contradiction. ☐

In Chapter 5, we will prove that the continuity assumption in Definition 2.2.11 is not necessary. Indeed, if a locally compact group has property (T), then any *non-continuous* representation with almost invariant vectors (as in 2.2.11) necessarily has an invariant vector. Moreover we will develop property (T) for bornological groups (see Section 2.1.3).

2.3 Amenability

We introduce amenability for locally compact groups. We will use it later in Chapter 6. Since it is not a central topic in this thesis we only state some important results and refer to [49] for details.

Let G be a locally compact group. Then there exists a non-trivial, regular, left-invariant measure μ on the Borel sigma-algebra of G. It is unique up to scalar multiplication and it is called the (left) Haar measure. For $p \in [1, \infty]$ we write $L^pG = L^p(G, \mu)$. The group G acts isometrically on these spaces by left-translation. For $p = 2$ this is the left-regular representation.

Definition 2.3.1. Let G be a locally compact group. A *mean* on $L^\infty G$ is a positive linear map $\varphi\colon L^\infty G \to \mathbb{C}$ satisfying $\varphi(1) = 1$. It is an *invariant mean* if it satisfies $\varphi(\lambda_g f) = \varphi(f)$ for all $f \in L^\infty G$ and $g \in G$, where λ_g denotes left-translation by g. The group G is *amenable* if there exists an invariant mean on $L^\infty G$.

There are also many other characterisations of amenable groups. An important one is the existence of Følner sets.

Definition 2.3.2. Let G be a locally compact group, let $K \subseteq G$ be a compact subset and let $\varepsilon > 0$. A (K, ε)-Følner set is a measurable subset $U \subseteq G$ with $0 < \mu(U) < \infty$ satisfying $\frac{\mu(KU \Delta U)}{\mu(U)} < \varepsilon$. Here Δ denotes the symmetric difference.

Now we have the following well-known result. We do not give the proof here, but a proof of a more general result can be found in Chapter 6.

Proposition 2.3.3. *Let G be a locally compact group with left Haar measure μ. The following are equivalent.*

(i) G is amenable.

(ii) For every compact $K \subseteq G$ and every $\varepsilon > 0$ there is a (K, ε)-Følner set.

(iii) The left-regular representation of G on L^2G has almost invariant vectors (see Definition 2.2.11).

Proof. See Proposition 6.3.1, applied to the measure-preserving action of G on (G, μ). □

All abelian groups are amenable. More generally, all solvable groups are amenable. Any group containing the free group F_2 as a subgroup is not amenable.

Proposition 2.3.4. *A non-compact amenable group does not have property (T).*

Proof. By Proposition 2.3.3(iii) the left-regular representation of G on L^2G has almost invariant vectors. An invariant vector in L^2G would correspond to a non-zero constant function on G, which would not be square-integrable because G is not compact. Therefore there are no invariant vectors and G does not have property (T). □

2.4 Geometric property (T)

2.4.1 Expander sequences

Let $X = (V, E)$ be a finite simple graph. This means that V is a finite set and E is a set of pairs $\{x, y\}$ with $x \neq y \in V$. We will write $x \sim y$ as a shorthand for $\{x, y\} \in E$. To this combinatorial object we can associate a matrix called the Laplacian Δ.

Definition 2.4.1. Let $X = (V, E)$ be a finite simple graph. The *Laplacian* is the matrix Δ, whose rows and columns are indexed by the elements of V, satisfying

$$\Delta_{xy} = \begin{cases} \deg(x) & \text{if } x = y; \\ -1 & \text{if } x \sim y; \\ 0 & \text{otherwise.} \end{cases}$$

Here $\deg(x)$ denotes the degree of vertex x.

Clearly the graph can be recovered from its Laplacian, since $\{x, y\} \in E$ if and only if $\Delta_{xy} = -1$. Note that the sum of every row is zero and the sum of every column is zero. The Laplacian Δ acts on the vector space $\mathbb{R}[V] = \mathbb{R}^V$. For a function $v \in \mathbb{R}^V$ we have $(\Delta v)(x) = \sum_{y\sim x} v(x) - v(y)$. We get

$$\langle \Delta v, v\rangle = \sum_x \overline{(\Delta v)(x)} v(x) = \sum_x \sum_{y\sim x} \overline{v(x) - v(y)} v(x) = \sum_{\{x,y\}\in E} |v(x) - v(y)|^2.$$

This quantity measures the variance of v on edges. We see that Δ is a positive semi-definite operator. Moreover $\Delta v = 0$ exactly when v is constant on each component of V. We denote the eigenvalues of Δ by $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{|V|-1}$ (with multiplicities). Then $\lambda_1 > 0$ exactly when the graph is connected. The value of λ_1 is related to a combinatorial feature of the graph, called the Cheeger constant.

Definition 2.4.2. Let (V, E) be a finite simple graph. For $A \subseteq V$ let ∂A be the set of edges with exactly one vertex in A. The *Cheeger constant* of the graph is

$$h = \inf\left\{\frac{|\partial A|}{|A|} \mid A \subseteq V, 0 < |A| \leq \frac{1}{2}|V|\right\}.$$

The following inequality is called Cheeger's inequality and was shown in [12].

Proposition 2.4.3. *Let (V, E) be a finite simple connected graph. Suppose the degree of every vertex is at most d. Let λ_1 be the smallest positive eigenvalue of the Laplacian and let h be the Cheeger constant. We have*

$$\frac{h^2}{2d} \leq \lambda_1 \leq 2h.$$

□

Now we will consider sequences of finite graphs $X_n = (V_n, E_n)$. We will assume that all of the X_n are finite simple connected graphs, and that $\lim_n |V_n| = \infty$. Moreover, we assume that the sequence has *uniformly bounded degree*, i.e. there is a constant d such that for all n, all vertices of X_n have degree at most d.

Definition 2.4.4. Let (X_n) be a sequence of connected graphs with uniformly bounded degree and number of vertices tending to infinity. We say that (X_n) is an *expander sequence* if there is a constant $c > 0$ such that for all n, all positive eigenvalues of the Laplacian of X_n are at least c.

By Proposition 2.4.3, we see that X_n is an expander sequence if and only if there is a constant $c' > 0$ such that the Cheeger constant of all X_n is at least c'. So we have both an algebraic and a combinatorial characterisation of expander sequences. We can see expander sequences as sequences of graphs which are well connected.

Explicit examples of expander sequences can be constructed from Cayley graphs. Let Γ be a group and S a symmetric subset of Γ, i.e. a subset satisfying $s \in S$ if and only if $s^{-1} \in S$. Then the *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ is the graph with vertex set equal to Γ, where different vertices g and h are connected if $g^{-1}h \in S$.

Example 2.4.5. Let $k \geq 3$ and consider $\mathrm{SL}_k(\mathbb{Z})$, the group of $k\times k$-matrices with integer coefficients and determinant one. Let $S \subseteq \mathrm{SL}_k(\mathbb{Z})$ be a finite, symmetric generating set. For an integer n let X_n be the Cayley graph $\mathrm{Cay}(\mathrm{SL}_k(\mathbb{Z}/n\mathbb{Z}), \overline{S})$, where $\overline{S}$ denotes the image of S in $\mathrm{SL}_k(\mathbb{Z}/n\mathbb{Z})$. Then the sequence (X_n) is an expander sequence (see [28, Theorem 4.3.1]).

Example 2.4.6. We can do a similar construction for $k = 2$. Let S be a finite, symmetric generating set for $\mathrm{SL}_2(\mathbb{Z})$. For a prime number p let X_p be the Cayley graph $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}), \overline{S})$. Then the sequence (X_p) is an expander sequence, where p ranges over all prime numbers (see [28, Theorem 6.1.1]). This example is considerably more technical to prove than the previous one.

We refer to e.g. [32] for more details on expanders.

2.4.2 Geometric property (T)

In this section $(X_n) = ((V_n, E_n))$ is a sequence of finite, simple, connected graphs satisfying $\lim_n |V_n| = \infty$ and for all n, every vertex of X_n has degree at most some natural number d. We denote the union of all these graphs by X and the union of the vertex sets by V. For $x, y \in V_n$ let $d(x, y)$ denote the length of the shortest path in X_n connecting x and y. Let $\ell^2 V_n$ denote the Hilbert space of functions on V_n.

Definition 2.4.7. Let $T_n \in B(\ell^2 V_n)$ be a matrix. The *propagation* of T_n is defined as

$$\mathrm{prop}(T_n) = \sup\{d(x, y) \mid x, y \in V_n,\ (T_n)_{x,y} \neq 0\}.$$

We denote the matrix norm of T_n by $\|T_n\|$. Let $\ell^2 V$ denote the Hilbert space of square-summable functions on V. If we have a sequence of matrices $T_n \in B(\ell^2 V_n)$ with $\sup_n \|T_n\| < \infty$ we can combine them into an operator $T = \prod_n T_n \in B(\ell^2 V)$, with norm $\|T\| = \sup_n \|T_n\|$.

Definition 2.4.8. Let $T = (T_n) \in \prod_n B(\ell^2 V_n) \cap B(\ell^2 V)$. The propagation of T is $\mathrm{prop}(T) = \sup_n \mathrm{prop}(T_n)$. If $\mathrm{prop}(T) < \infty$ we say that T is a *finite propagation operator*. We denote by $\mathbb{C}_{\mathrm{cs}}[X]$ the set of all finite propagation operators.

Note that if S and T are finite propagation operators, then $\mathrm{prop}(S+T) \leq \max(\mathrm{prop}(S), \mathrm{prop}(T))$ and $\mathrm{prop}(ST) \leq \mathrm{prop}(S) + \mathrm{prop}(T)$ and $\mathrm{prop}(T^*) = \mathrm{prop}(T)$. Thus $\mathbb{C}_{\mathrm{cs}}[X]$ is a $*$-subalgebra of $B(L^2 V)$ (it is not closed). It is called the *algebraic uniform Roe algebra* (introduced in [43]).

Let $\Delta_n \in B(L^2 V_n)$ denote the Laplacian of X_n and let $\Delta = \prod_n \Delta_n$. Then we have $\mathrm{prop}(\Delta) = 1$, so Δ is an element of $\mathbb{C}_{\mathrm{cs}}[V]$. The spectrum of Δ in the C^*-algebra $B(\ell^2 V)$ is the closure of the unions of the spectra of the Δ_n. Thus we see that the X_n form an expander sequence if and only if there is $c > 0$ such that $\sigma(\Delta) \subseteq \{0\} \cup [c, \infty)$.

This characterisation of expander sequences considers the spectrum of Δ in the C^*-algebra $B(\ell^2 V)$, so it does not make use of the algebra $\mathbb{C}_{\mathrm{cs}}[X]$. Since this subalgebra is not closed, we can equip it with the *maximal norm*, which we explain now.

Definition 2.4.9. A *representation* of $\mathbb{C}_{\mathrm{cs}}[X]$ consists of a Hilbert space $\mathcal{H}$ and a $*$-morphism $\pi\colon \mathbb{C}_{\mathrm{cs}}[X] \to B(\mathcal{H})$. The *maximal norm* on $\mathbb{C}_{\mathrm{cs}}[X]$ is given by $\|T\|_{\max} = \sup_{(\pi,\mathcal{H})} \|\pi(T)\|$, where the supremum ranges over all representations.

It turns out that the maximal norm takes only finite values (see [19, Section 3] or Corollary 4.6.4). From this it is easy to see that it indeed defines a norm on $\mathbb{C}_{\mathrm{cs}}[X]$.

Example 2.4.10. The inclusion $\mathbb{C}_{\mathrm{cs}}[X] \hookrightarrow B(\ell^2 V)$ defines a representation.

We denote the completion of $\mathbb{C}_{\mathrm{cs}}[X]$ with respect to the maximal norm by $C^*_{\max}(X)$ (see e.g. [56]). The spectrum of Δ in this algebra is denoted by $\sigma_{\max}(\Delta)$. It contains $\sigma(\Delta)$.

Definition 2.4.11. Let (X_n) be a sequence of finite graphs with bounded degree and number of vertices going to infinity. The sequence has *geometric property (T)* if there is a constant $c > 0$ such that $\sigma_{\max}(\Delta) \subseteq \{0\} \cup [c, \infty)$.

This is strictly stronger than being an expander. The following proposition is from [58]. It relates geometric property (T) with property (T) for groups (see Section 2.2).

Proposition 2.4.12. *Let Γ be a finitely generated group and let S be a finite, symmetric set of generators of Γ. Let $N_1 \supseteq N_2 \supseteq \cdots$ be a sequence of normal, finite-index subgroups of Γ satisfying $\bigcap_n N_n = \{1\}$. Consider the sequence of Cayley graphs $X_n = \mathrm{Cay}(\Gamma/N_n, S/N_n)$. This sequence has geometric property (T) if and only if the group Γ has property (T).*

Proof. See [58, Theorem 7.3], or Theorem 5.H for a slightly more general result. □

Example 2.4.13. Let $k \geq 2$ and let S be a finite, symmetric generating set for $\mathrm{SL}_k(\mathbb{Z})$. For prime numbers p let $X_p = \mathrm{Cay}(\mathrm{SL}_k(\mathbb{F}_p), \overline{S})$ where $\overline{S}$ denotes the image of S in $\mathrm{SL}_k(\mathbb{F}_p)$. Then the sequence (X_p) has geometric property (T) if and only if $k \geq 3$ (see Example 2.2.8). Compare this with Examples 2.4.5 and 2.4.6, which say that the X_p form an expander sequence for all $k \geq 2$.

We only have an algebraic definition for geometric property (T). It would be very interesting to have a combinatorial characterisation as well. This is not known to exist, but there are some combinatorial properties that all graph sequences with property (T) share. For example, for any such a sequence there is an integer R such that all but finitely many of the graphs contain an R-cycle. This is proven in [57, Corollary 7.4]. In Chapter 3 we generalise this result; indeed, we will prove that there is an integer R such that all but finitely many of the graphs have a *lot* of R-cycles (see Theorem 3.A).

If (X_n) is a sequence of graphs with geometric property (T), and we modify this sequence of graphs by adding or removing a small number of edges and/or vertices to get a new sequence (Y_n), then the new sequence need not have geometric property (T); indeed, it may not be an expander. However in Chapter 3 we show that if (Y_n) still is an expander sequence, then it also still satisfies geometric property (T). See Theorem 3.B for more details.

A sequence of graphs can be seen as a metric space (using the shortest-path distance). Then we can also see it as a coarse space (see Section 2.1). In [58, Proposition 4.6] it is shown that geometric (T) is invariant under coarse equivalences.

Proposition 2.4.14. *Let (X_n) and (Y_n) be sequences of graphs, whose number of vertices go to infinity, and with bounded degree. Assume that the coarse spaces $X = \bigsqcup_n X_n$ and $Y = \bigsqcup_n Y_n$ are coarsely equivalent. Then X has geometric property (T) if and only if Y has geometric property (T).*

□

Geometric property (T) can also be defined directly for coarse spaces, without making use of a graph structure (see [58]). The coarse space should then be assumed to be *uniformly locally finite*: this means that for every controlled set $E \subseteq X \times X$ there is an integer N such that $\#\{y \mid (x, y) \in E\} \leq N$ for all $x \in X$. Still if X and Y are coarsely equivalent spaces satisfying this condition, X has geometric property (T) if and only if Y has geometric property (T). However, being locally finite is not invariant under coarse equivalence, so this needs to be assumed for both X and Y.

In Chapter 4 we study the more general setting of coarse spaces with *bounded geometry*. A coarse space has bounded geometry if and only if it is coarsely equivalent to a uniformly locally finite coarse space. In particular, bounded geometry is invariant under coarse equivalences. We develop geometric property (T) for all coarse spaces with bounded geometry, and we show that it is invariant under coarse equivalences (see Theorem 4.A).

2.5 Embeddings of sequences of graphs into Hilbert spaces

Let (X_n) be a sequence of graphs with uniformly bounded degree such that the number of vertices of X_n converges to infinity. We will identify the graphs with their vertex sets and view them as a metric space using the shortest path distance. Let X be the union of all X_n. We can view X as a metric space, where two vertices in different elements of the sequence have distance infinity.

Let (V, d) be another metric space. We call a map $F\colon X \to V$ a *coarse embedding* if there is a constant L and an increasing unbounded map $\rho\colon \mathbb{N} \to [0, \infty)$ such that for all n and $x, y \in X_n$ we have $d(F(x), F(y)) \leq L$ if x and y are adjacent, and $d(F(x), F(y)) \geq \rho(d(x, y))$, where $d(x, y)$ denotes the shortest path distance between x and y.

It is an interesting question to ask whether a certain sequence of graphs can be embedded into a Hilbert space. It turns out that this is impossible for expander graphs (see for example [44, Proposition 11.29]).

Lemma 2.5.1. *Let (X_n) be an expander sequence. Then there is no coarse embedding of $X = \bigsqcup X_n$ into a Hilbert space $\mathcal{H}$.*

Proof. Let Δ_n denote the Laplacian of X_n and let $c > 0$ be such that $\sigma(\Delta_n) \subseteq \{0\} \cup [c, \infty)$ for all n. Let d be the maximal degree of all vertices of all X_n. Then for all functions $f\colon X_n \to \mathbb{C}$ with $\sum_{x \in X_n} f(x) = 0$ we have $\langle \Delta f, f\rangle \geq c\|f\|^2$. We also have $\langle \Delta f, f\rangle = \frac{1}{2}\sum_{(x,y)\in E(X_n)} |f(x) - f(y)|^2$ and $\|f\|^2 = \sum_{x\in X_n} |f(x)|^2 = \frac{1}{2|X_n|}\sum_{x,y\in X_n} |f(x) - f(y)|^2$. So we get

$$\sum_{(x,y)\in E(X_n)} |f(x) - f(y)|^2 \geq \frac{c}{|X_n|}\sum_{x,y\in X_n} |f(x) - f(y)|^2.$$

Since neither side changes when we add a constant to f, this inequality holds for all functions $f\colon X_n \to \mathbb{C}$.

Now suppose there is a coarse embedding $F\colon X \to \mathcal{H}$. Let (e_i) be an orthonormal basis for $\mathcal{H}$. Then for all n we have

$$\begin{aligned}\sum_{(x,y)\in E(X_n)} \left\|F(x) - F(y)\right\|^2 &= \sum_i \sum_{(x,y)\in E(X_n)} |\langle F(x) - F(y), e_i\rangle|^2 \\ &\geq \frac{c}{|X_n|}\sum_i \sum_{x,y\in X_n} |\langle F(x) - F(y), e_i\rangle|^2 \\ &= \frac{c}{|X_n|}\sum_{x,y\in X_n} \left\|F(x) - F(y)\right\|^2.\end{aligned}$$

Since the function F is L-Lipschitz for some L, the left-hand side is at most $d|X_n|L^2$. At the same time, since F is a coarse embedding, there is some R such that if $d(x, y) \geq R$ then $\left\|F(x) - F(y)\right\| \geq 2\sqrt{\frac{d}{c}}L$. Moreover, for large enough n, more than half of the pairs $(x, y) \in X_n^2$ have a distance more than R. Thus the right-hand side is at least $2d|X_n|L^2$, a contradiction. $\square$

More generally, an expander sequence cannot be coarsely embedded into any ℓ^p-space for $1 \leq p < \infty$ (see [36, Equation 6]). Clearly then, any graph sequence that allows a coarse embedding of an expander sequence can also not be coarsely embedded into any ℓ^p-space. It is also known that, for $p > 2$, a Hilbert space can be coarsely embedded into an ℓ^p-space but an ℓ^p-space does not embed into a Hilbert space (see [40] and [25]). Thus coarse embeddability in ℓ^p is in general weaker than coarse embeddability in a Hilbert space. Hence, a natural question is: does there exist a sequence of graphs with uniformly bounded degree that coarsely embeds into some ℓ^p space, for $2 < p < \infty$, but not into ℓ^2?

This would necessarily be a graph sequence that contains no coarsely embedded expander. In [2, Theorem 7.1], as well as in [10], graph sequences with uniformly bounded degree are constructed that cannot embed into a Hilbert space, and also contains no coarsely embedded expander sequence. However they do not coarsely embed into an ℓ^p-space either.

We explore this question in Chapter 7. The approach we take there requires a model of random graphs embedded in the torus $\mathbb{T}^m$. Since no suitable model is known, we develop a model of random graphs embedded in a manifold. This model is described in Chapter 8. Unfortunately we were only able to accurately describe such random graphs embedded in the manifold $\mathbb{R}^m$. Thus the question above remains open.

3 Cycles in graphs with geometric property (T)

This chapter is based on the article [55]. Large parts of it are reproduced verbatim.

Chapter summary

We show that a sequence of graphs with uniformly bounded vertex degrees, number of vertices going to infinity, and with geometric property (T) has many small cycles. We also show that when a small part of such a sequence of graphs with geometric property (T) is changed, it still has geometric property (T), provided that it is still an expander. We use this to give an example of a sequence of graphs with geometric property (T) that has large cycle-free balls.

3.1 Introduction and main results

A sequence of finite graphs (X_n) has *large girth* if for every $R > 0$ there is a positive integer N such that for $n \geq N$, the graphs X_n do not have any cycles of length shorter than R. It was shown in [57, Corollary 7.5] that a sequence of graphs with geometric property (T) can never have large girth, using properties of K-theory. We give a quantitative version of this result: there is some R such that all of the X_n have "many" R-cycles.

Let us make this more precise. As in [13], we define the cycle spaces of a graph.

Definition 3.1.1. Let X be a graph with edge set E. Let $\mathbb{C}[E]$ be the free $\mathbb{R}$-vector space generated by E. We use the convention that $(x, y) = -(y, x)$. Let $Z(X)$ be the subspace of $\mathbb{C}[E]$ generated by the cycles, where a cycle $(x_1, x_2, \ldots, x_n)$ corresponds to the element $(x_1, x_2) + (x_2, x_3) + \ldots + (x_{n-1}, x_n) + (x_n, x_1)$. For any integer R, let $Z_R(X)$ be the subspace generated by the cycles of length at most R.

We can now state the first theorem of this chapter.

Theorem 3.A. Suppose $X = (X_n)$ is a sequence of finite connected graphs with uniformly bounded degree and with number of vertices going to infinity. Suppose that X has geometric property (T). Then there are constants $R > 0$ and $\varepsilon > 0$ such that for all large enough n, we have $\dim Z_R(X_n) \geq \varepsilon |V(X_n)|$.

The *cost* of generating an equivalence relation is introduced by Levitt in [30]. Taking the supremum of all equivalence relations generated by probability measure preserving actions of a group gives the cost of this group, as introduced by Gaboriau in [16]. *Combinatorial cost* is a variant on this concept for sequences of graphs which was defined by Elek in [13]. It measures the number of edges necessary to induce the coarse structure of a sequence of graphs.

Definition 3.1.2. Let (X_n) and (Y_n) be sequences of finite connected graphs, such that X_n and Y_n have the same vertex set. For $x, y \in V(X_n)$ denote by $d_X(x, y)$ the path distance in X_n, and by $d_Y(x, y)$ the path distance in Y_n. We say that (X_n) and (Y_n) *induce the same coarse structure* if there is a constant L such that $d_X(x, y) \leq L d_Y(x, y)$ and $d_Y(x, y) \leq L d_X(x, y)$ for all n and $x, y \in V(X_n)$.

Definition 3.1.3 ([13])**.** The *combinatorial cost* of a sequence $X = (X_n)$ of graphs is defined as

$$c(X) = \inf_Y \liminf_n \frac{|E(Y_n)|}{|V(X_n)|}$$

where the infimum is taken over all sequences of graphs (Y_n) on the same vertex sets as X_n, which induce the same coarse structure as (X_n).

In [24], it was shown that any group with property (T) has cost 1. This raises the question if a similar theorem can be proved in the combinatorial context. From Theorem 3.A it already follows that the infimum in Definition 3.1.3 is not attained for sequences of graphs with geometric property (T) (see Proposition 3.2.7). The question, whether sequences of graphs with geometric property (T) necessarily have cost 1, remains open.

Theorem 3.A also raises the following question: is it true that for every sequence (X_n) with geometric property (T), there is an R such that all vertices have a cycle in their R-neighbourhood? We show that if two graph sequences (X_n) and (Y_n) "mostly" agree, and one of them has geometric property (T) and the other one is an expander, then the other one has geometric property (T) too. This can be used to answer the above question negatively (see Corollary 3.3.2).

Definition 3.1.4. Let $X = (X_n)$ and $Y = (Y_n)$ be sequences of finite connected graphs with increasing number of vertices and uniformly bounded degree. We say that X and Y are *approximately isomorphic* if there are subgraphs $X'_n \subseteq X_n$ and $Y'_n \subseteq Y_n$ such that X'_n and Y'_n are isomorphic for all n, and

$$\lim_{n\to\infty} \frac{|V(X'_n)|}{|V(X_n)|} = \lim_{n\to\infty} \frac{|E(X'_n)|}{|E(X_n)|} = \lim_{n\to\infty} \frac{|V(Y'_n)|}{|V(Y_n)|} = \lim_{n\to\infty} \frac{|E(Y'_n)|}{|E(Y_n)|} = 1.$$

It is easy to see that we may take X'_n and Y'_n to be induced subgraphs (that is, subgraphs that contain all edges between their vertices that are present in the ambient graph).

If X and Y are approximately isomorphic and X has geometric property (T), then Y need not have geometric property (T); indeed, it does not even need to be an expander (it is only an asymptotic expander, as defined in [27]). For example, if (X_n) is a sequence of graphs with geometric property (T) with $|X_n| \geq n^2$, and we attach a path of length n to each X_n, then it will not be an expander anymore. However, this is the only thing that can go wrong.

Theorem 3.B. Let $X = (X_n)$ and $Y = (Y_n)$ be approximately isomorphic sequences of finite connected graphs with uniformly bounded degree and number of vertices going to infinity. Suppose X has geometric property (T) and Y is an expander. Then Y has geometric property (T).

3.2 Short-cycle spaces in graphs with geometric property (T)

Let (X_n) be a sequence of graphs. The algebraic Roe algebra $\mathbb{C}_{\mathrm{cs}}[X]$, as recalled in Section 1, has a standard representation on $\ell^2 X$. Other representations are harder to describe explicitly. We will see how to construct a representation of $\mathbb{C}_{\mathrm{cs}}[X]$, starting from a representation of only part of the matrix algebra $B(\ell^2 X_n)$, for each n.

Definition 3.2.1. Let X be a graph and let $B_R(\ell^2 X)$ be the set of bounded operators on $\ell^2 X$ of propagation at most R. An R-representation is a Hilbert space $\mathcal{H}$ together with a linear map $\pi\colon B_R(\ell^2 X) \to B(\mathcal{H})$ satisfying $\pi(T^*) = \pi(T)^*$ and $\pi(TS) = \pi(T)\pi(S)$ if T, S and TS have propagation at most R.

Suppose $X = (X_n)$ is a sequence of graphs and suppose we have a sequence of r_n-representations π_n of X_n on $\mathcal{H}_n$. If $r_n \to \infty$, we can make a representation of $\mathbb{C}_{\mathrm{cs}}[X]$ as follows: let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and define the ultraproduct $\mathcal{H} = \prod_n \mathcal{H}_n/\mathcal{U}$. Define $\pi(T) = \lim_{\mathcal{U}} \pi_n(T|_{\ell^2 X_n}) \in B(\mathcal{H})$. Then $\pi\colon \mathbb{C}_{\mathrm{cs}}[X] \to B(\mathcal{H})$ is a representation.

We can use these representations to prove that a sequence of graphs with property (T) does not have large girth (which was also already proved in [57] using K-theory). Below we first give the proof below for d-regular graphs with an even integer d, to avoid technicalities. It is true in greater generality; this will follow later from Theorem 3.A.

Proposition 3.2.2. *Let $X = (X_n)$ be a sequence of d-regular finite connected graphs and an increasing number of vertices. Suppose that X has geometric property (T) and that d is even. Then X does not have large girth.*

Proof. Suppose that X has large girth and let $t \in \mathbb{R}$. Let r_n denote the girth of X_n. There is an Eulerian cycle on X_n. This defines a direction on each edge of X_n. So we can define a function $\rho\colon E_n \to \{-1, 1\}$, where E_n denotes the edge set of X_n, such that for each $x \in X_n$ we have $\sum_{y\sim x} \rho(x, y) = 0$. For all pairs $(x, y) \in X_n^2$ choose a shortest path $\gamma_{(x,y)} = (x = x_0, x_1, \ldots, x_{d(x,y)} = y)$ from x to y. Choose it in such a way that $\gamma_{(y,x)}$ is the inverse of $\gamma_{(x,y)}$. Define $\rho\colon X_n^2 \to \mathbb{Z}$ by $\rho(x, y) = \rho(x_0, x_1) + \ldots + \rho(x_{d(x,y)-1}, x_{d(x,y)})$. Then $\rho(x, y) = -\rho(y, x)$, and if $d(x, y) + d(y, z) + d(x, z) < r_n$, we have $\rho(x, y) + \rho(y, z) = \rho(x, z)$. Now define $\pi_n\colon B(\ell^2 X_n) \to B(\ell^2 X_n)$ by

$$\pi_n(T)\xi(x) = \sum_y T_{xy} \exp(it\rho(x, y))\xi(y).$$

Then $\pi_n(T^*) = \pi_n(T)^*$, and if the propagations of S and T and TS are at most $\frac{1}{3}r_n$, then $\pi_n(TS) = \pi_n(T)\pi_n(S)$. So π_n defines a $\frac{1}{3}r_n$-representation.

Consider the constant unit vector $\xi_n \in \ell^2 X_n$ with $\xi_n(x) = \frac{1}{\sqrt{|V(X_n)|}}$. This satisfies $\pi_n(\Delta_X)\xi_n = d(1 - \cos t)\xi_n$. Let $\pi\colon \mathbb{C}_{\mathrm{cs}}[X] \to B(\mathcal{H})$ denote the limit of the π_n. This is a representation. Let $\xi = (\xi_n) \in \mathcal{H}$. Then $\pi(\Delta_X)\xi = d(1 - \cos t)\xi$, so $d(1 - \cos t) \in \sigma_{\max}(\Delta_X)$. Since t can be any real number, we conclude that $[0, 2d] \subseteq \sigma_{\max}(\Delta_X)$. Then X does not have geometric property (T), and we have arrived at a contradiction. □

If the graph is not d-regular for an even d, we do not necessarily have a Eulerian cycle. In this case, we can still prove the result. We need Lemmas 3.2.3 and 3.2.4 below to construct a cycle $v \in Z(X)$ that will play a similar role as the Eulerian cycle above. We will use it to prove something even stronger, namely that there are "many" small cycles in X (Theorem 3.A).

Note that, if X is a finite connected, then $\dim(\mathbb{C}[E]) = |E|$ and $\dim(Z(X)) = |E|-|X|+1$ (this is clear for a tree, and any time an edge is added to a graph, both sides increase by 1). We will also use the standard inner product on $\mathbb{C}[E]$, such that edges have norm 1 and are perpendicular to each other. An edge in a connected graph is called a *bridge* if the removal of this edge would render the graph disconnected.

Lemma 3.2.3. *Let X be a finite connected graph with edge set E. There is $v \in Z(X)$ satisfying the following conditions:*

(i) For all $e \in E$ we have $v(e) \in \{-1, 0, 1\}$.

(ii) Of all edges $e \in E$ that are not bridges, at most half satisfy $v(e) = 0$.

Proof. Let $Z_1, \ldots, Z_K$ be cycles in X such that each edge that is not a bridge is contained in one of the cycles Z_i. Choose $\varepsilon_1, \ldots, \varepsilon_K \in \{0, 1\}$ uniformly and independently at random. Let $w = \sum_{i=1}^K \varepsilon_i Z_i$. For each $e \in E$ that is not a bridge, we have $\mathbb{P}[w(e) \text{ is odd}] = \frac{1}{2}$. So $\mathbb{E}[\#\{e \in E \mid$

$w(e)$ is odd$\}]$ is equal to half the number of edges that are not bridges. So there is some $w \in Z(X)$ such that $w(e)$ is odd for at least half the number of edges that are not bridges. Now let $E' \subseteq E$ be the subset of E consisting of the edges for which $w(e)$ is odd. Then each vertex of X has an even number of adjacent edges in E'. So each component of $(V(X), E')$ has a Eulerian cycle. Let v be the sum of these Eulerian cycles. Then v satisfies the conditions. $\square$

Lemma 3.2.4. *Let X be a finite connected graph with edge set E and maximal degree d. Suppose there is a constant $h > 0$ such that for all subsets A with $\frac{1}{4}|V(X)| \leq |A| \leq \frac{1}{2}|V(X)|$, we have $|\partial A| \geq h|A|$. Then there is a constant $c > 0$, only depending on d and h, such that if $|V(X)|$ is large enough, the number of edges that are not bridges in X is at least $c|E|$.*

Proof. For b a bridge, define K_b and G_b to be the two components of the graph $(V(X), E \setminus \{b\})$, with $|K_b| \leq |G_b|$. Since $\partial K_b = \{b\}$, and $h \cdot \frac{1}{4}|V(X)| > 1$ for $|V(X)|$ large enough, we must have $|K_b| < \frac{1}{4}|V(X)|$. Now let $G = \cap_b G_b$ and $K = V(X) \setminus G$, where the intersection ranges over all bridges b. If $x, y \in G$, then all the vertices of any minimal path from x to y are also in G. In particular, G is connected.

We show that G is non-empty. Let b be a bridge such that K_b is maximal. Let x be the endpoint of b in G_b. Then $x \in G$: for if there is another bridge b' with $x \in K_{b'}$, we have either $K_b \cup \{x\} \subseteq K_{b'}$, or $G_b \subseteq K_{b'}$, and then $K_{b'}$ is too large.

Note that all bridges of X must have at least one vertex in K. Therefore the number of edges that are not bridges is at least $|G| - 1$. So if we find a constant $c' > 0$ such that $|G| \geq c'|V(X)|$, we are done.

Let $\partial G = \{b_1, \ldots, b_N\}$. These are all bridges. For each $1 \leq i \leq N$ we have $|K_{b_i}| \leq \frac{1}{4}|V(X)|$. The K_{b_i} are pairwise disjoint: if K_{b_i} and K_{b_j} were not disjoint for $i \neq j$, then we would find a cycle with b_i and b_j. Then we can choose some $1 \leq M \leq N$ such that $\min(|K|, \frac{1}{4}|V(X)|) \leq \sum_{j=1}^{M} |K_{b_j}| \leq \frac{1}{2}|V(X)|$. If $|K| < \frac{1}{4}|V(X)|$, we have $|G| \geq \frac{3}{4}|V(X)|$, and we are done. Suppose that $|K| \geq \frac{1}{4}|V(X)|$. Then we can apply the assumption of the lemma to $A = \cup_{j=1}^{M} K_{b_j}$. We have $|\partial A| = M \leq N = |\partial G| \leq d|G|$, therefore $|A| \leq \frac{d}{h}|G|$. So $|G| \geq \frac{h}{4d}|V(X)|$. This finishes the proof. $\square$

Corollary 3.2.5. *Let X be a finite connected graph with edge set E and maximal degree d. Suppose there is a constant $h > 0$ such that for all subsets $A \subset V(X)$ with $\frac{1}{4}|V(X)| \leq |A| \leq \frac{1}{2}|V(X)|$, we have $|\partial A| \geq h|A|$. Then, provided that $|V(X)|$ is large enough, there are a constant $c > 0$, only depending on d and h, and $v \in Z(X)$ satisfying the following conditions:*

(i) For all $e \in E$ we have $v(e) \in \{-1, 0, 1\}$.

(ii) We have $\#\{e \in E \mid v(e) \neq 0\} \geq c|E|$.

Proof. This follows directly from Lemmas 3.2.3 and 3.2.4. $\square$

Theorem 3.A. Suppose $X = (X_n)$ is a sequence of finite connected graphs with uniformly bounded degree and with number of vertices going to infinity. Suppose that X has geometric property (T). Then there are constants $R > 0$ and $\varepsilon > 0$ such that for all large enough n, we have $\dim Z_R(X_n) \geq \varepsilon|V(X_n)|$.

Proof. Let $\gamma > 0$ be such that $\sigma_{\max}(\Delta_X) \subseteq \{0\} \cup [\gamma, \infty)$. Let d be the maximal degree of all vertices of X. Since X has geometric property (T), it is in particular an expander sequence by Cheeger's inequality: for each subset $A \subseteq X_n$ with $|A| \leq \frac{1}{2}|V(X_n)|$ we have $|\partial A| \geq \frac{\gamma}{2}|A|$. Only finitely many of the X_n can be trees (this follows from Lemma 3.2.4, for example). We can then assume without loss of generality that none of the X_n is a tree.

Define $h = \frac{\gamma}{4}$. Let c_1 be the constant from Corollary 3.2.5, using the constants d and h. Let $c_2 > 0$ be such that

$$c_3 = \frac{1}{4}d^2 - \left(1 - \frac{c_1}{2d}\right)\left(\frac{1}{2}d + c_2 d\right)^2 - \frac{c_1}{2d}\left(\frac{1}{2}d + c_2 d - \frac{1}{2}\right)^2 > 0. \tag{1}$$

Let $t > 0$ be small enough such that

$$dt^2 < \gamma \tag{2}$$

and also

$$\left|\exp(it) - 1 - it + \frac{1}{2}t^2\right| \leq c_2 t^2. \tag{3}$$

Let $\varepsilon > 0$ be such that the following conditions are satisfied:

$$8\varepsilon d^2 \leq \frac{1}{2}c_3 t^4, \tag{4}$$

$$2\varepsilon \leq \frac{c_1}{2d}, \tag{5}$$

$$4\varepsilon \leq h. \tag{6}$$

Suppose for a contradiction that for each R, there is an n such that $\dim Z_R(X_n) < \varepsilon|V(X_n)|$. Let E_n be the set of edges of X_n. Note that $Z(X_n) \subseteq \mathbb{C}[E_n]$ consists of the functions $\rho\colon E_n \to \mathbb{C}$ satisfying $\sum_{y\sim x}\rho(x,y) = 0$ for all $x \in V(X_n)$. The subset $Z_R(X_n)^\perp \subseteq \mathbb{C}[E_n]$ consists of all function $\rho\colon E_n \to \mathbb{C}$ satisfying $\rho(x_1,x_2) + \rho(x_2,x_3) + \ldots + \rho(x_q,x_1) = 0$ for all q-cycles $(x_1,x_2,\ldots,x_q)$ with $q \leq R$.

Let $\rho \in Z_R(X_n)^\perp$. We will construct a $\frac{1}{3}R$-representation π_ρ of X_n. First, we extend ρ to a function on $V(X_n)^2$ as follows: for each pair $(x,y) \in V(X_n)^2$, choose a shortest path $(x = x_0, x_1, \ldots, x_{d(x,y)} = y)$ from x to y and define $\rho(x,y) = \rho(x_0,x_1) + \rho(x_1,x_2) + \ldots + \rho(x_{d(x,y)-1}, x_{d(x,y)})$. Since $\rho \in Z_R(X_n)^\perp$, this satisfies the following: if $x,y \in V(X_n)$ satisfy $d(x,y) \leq R$, then $\rho(x,y) = -\rho(y,x)$, and if $x,y,z \in V(X_n)$ satisfy $d(x,y)+d(y,z)+d(z,x) \leq R$, then $\rho(x,y)+\rho(y,z)+\rho(z,x) = 0$.

Now we define $\pi_\rho : B(\ell^2 X_n) \to B(\ell^2 X_n)$ by

$$\pi_\rho(T)\xi(x) = \sum_y T_{xy}\exp(i\rho(x,y))\xi(y).$$

This is a $\frac{1}{3}R$-representation.

For a subset $B \subseteq E_n$, denote by $X_n \backslash B$ the graph $(X_n, E_n \backslash B)$. Choose edges $b_1, \ldots, b_{\dim(Z_R(X_n))}$ recursively in such a way that b_j is an edge in a cycle in $Z_R(X_n) \cap Z(X_n \setminus \{b_1,\ldots,b_{j-1}\})$. Then $Z_R(X_n) \cap Z(X_n \setminus \{b_1,\ldots,b_{j-1},b_j\})$ has one dimension fewer than $Z_R(X_n) \cap Z(X_n \setminus \{b_1,\ldots,b_{j-1}\})$. Thus, with $B = \{b_1,\ldots,b_{\dim(Z_R(X_n))}\}$ we have $Z_R(X_n) \cap Z(X_n \setminus B) = 0$. Then we also have $Z_R(X_n) \cap \mathbb{C}[E_n \setminus B] = 0$, and counting dimensions, $Z_R(X_n) \oplus \mathbb{C}[E_n \setminus B] = \mathbb{C}[E_n]$. Since $\mathbb{C}[B] \perp \mathbb{C}(E_n \setminus B)$ and $Z_R(X_n)^\perp \perp Z_R(X_n)$, it follows that $\mathbb{C}[B] \cap Z_R(X_n)^\perp = 0$. Counting dimensions again, we get $Z_R(X_n)^\perp \oplus \mathbb{C}[B] = \mathbb{C}[E_n]$.

Since each b_j is an edge in a cycle in $Z(X_n \setminus \{b_1,\ldots,b_{j-1}\})$, it follows by induction that $X_n \setminus B$ is still a connected graph. Let $A \subset V(X_n)$, with $\frac{1}{4}|V(X_n)| \leq |A| \leq \frac{1}{2}|V(X_n)|$. Denote by $\partial_{X_n\setminus B}A$

the set of edges in $X_n \setminus B$ with exactly one vertex in A. Then we have $|\partial_{X_n \setminus B} A| \geq |\partial_{X_n} A| - |B| \geq \frac{\gamma}{2}|A| - \varepsilon|V(X_n)| \geq (\frac{\gamma}{2} - 4\varepsilon)|A| = (2h - 4\varepsilon)|A| \geq h|A|$ by inequality (6). So $X_n \setminus B$ satisfies the conditions of Corollary 3.2.5. We can also assume $V(X_n)$ is large enough to apply the corollary, by taking R large enough.

Now let $v \in C(X_n \setminus B)$ be as in Corollary 3.2.5. Recall that $Z_R(X_n)^\perp \oplus \mathbb{C}[B] = \mathbb{C}[E_n]$. Hence there is a unique function $\rho \in Z_R(X_n)^\perp$ such that $\rho(e) = v(e)$ for all $e \notin B$.

Now we consider the $\frac{1}{3}R$-representation $\pi_{t\rho}$. Let $\Delta_t = \pi_{t\rho}(\Delta_X) \in B(\ell^2 X_n)$. Let $\xi_n \in \ell^2 X_n$ be the constant function with $\xi_n(x) = 1$ for all $x \in X_n$. Then we have

$$\Delta_t \xi_n(x) = \sum_{y \sim x} (1 - \exp(it\rho(x,y)))$$

for all $x \in X_n$.

We define a partition $V(X_n) = A_1 \cup A_2 \cup A_3$. We can think of the vertices in A_1 as the 'good' vertices, the vertices in A_2 as the 'neutral' vertices and the vertices in A_3 as the 'bad' vertices. Luckily, A_3 will be small. The sets are defined as follows:

$$\begin{aligned}
A_3 &= \{x \in X_n \mid (x,y) \in B \text{ for some } y \sim x\}, \\
A_1 &= \{x \in X_n \setminus A_3 \mid v(x,y) \neq 0 \text{ for some } y \sim x\}, \\
A_2 &= X_n \setminus (A_1 \cup A_3).
\end{aligned}$$

Since $|B| \leq \varepsilon|V(X_n)|$, we have $|A_3| \leq 2\varepsilon|V(X_n)|$, and since at least $c_1|E_n|$ edges e satisfy $v(e) \neq 0$, we have $|A_1| \geq \frac{c_1}{d}|E_n| - |A_3| \geq (\frac{c_1}{d} - 2\varepsilon)|V(X_n)| \geq \frac{c_1}{2d}|V(X_n)|$ by inequality (5).

We will estimate $|\Delta_t \xi_n(x) - \frac{1}{2}dt^2|$, finding increasingly good estimates in the cases $x \in A_3, A_2, A_1$ respectively. For all $x \in X_n$, we have $|\Delta_t \xi_n(x) - \frac{1}{2}dt^2| \leq 2d$. For $x \notin A_3$ we have $\rho(x,y) = v(x,y)$ for all $y \sim x$. Since $\sum_{y \sim x} v(x,y) = 0$, we have

$$\Delta_t \xi_n(x) = \sum_{y \sim x} (1 + itv(x,y) - \exp(itv(x,y))).$$

Then, using inequality (3) we have:

$$\begin{aligned}
\left| \Delta_t \xi_n(x) - \frac{1}{2}dt^2 \right| &\leq \sum_{y \sim x} \left| 1 + itv(x,y) - \frac{d}{2\deg(x)} t^2 - \exp(itv(x,y)) \right| \\
&\leq \left(\sum_{y \sim x} \left| 1 + itv(x,y) - \frac{d}{2\deg(x)} t^2 - (1 + itv(x,y) - \frac{1}{2}t^2 v(x,y)^2) \right| \right) + c_2 dt^2 \\
&\leq \frac{1}{2}dt^2 + c_2 dt^2 - \frac{1}{2}\sum_{y \sim x} v(x,y)^2 t^2.
\end{aligned}$$

For $x \in A_2$, this is at most $(\frac{1}{2}d + c_2 d)t^2$. For $x \in A_1$, there is $y \sim x$ with $v(x,y) \neq 0$, and we find that $|\Delta_t \xi_n(x) - \frac{1}{2}dt^2| \leq (\frac{1}{2}d + c_2 d - \frac{1}{2})t^2$.

Using inequalities (1) and (4), we now have:

$$\left\| \Delta_t \xi_n - \frac{1}{2}dt^2 \xi_n \right\|^2 = \sum_{x \in X_n} |\Delta_t \xi_n(x) - \frac{1}{2}dt^2|^2$$

$$
\begin{aligned}
&\leq \sum_{x\in A_1}\left(\frac{1}{2}d+c_2d-\frac{1}{2}\right)^2 t^4+\sum_{x\in A_2}\left(\frac{1}{2}d+c_2d\right)^2 t^4+\sum_{x\in A_3}4d^2\\
&\leq \left(\left(1-\frac{c_1}{2d}\right)\left(\frac{1}{2}d+c_2d\right)^2+\frac{c_1}{2d}\left(\frac{1}{2}d+c_2d-\frac{1}{2}\right)^2\right)|V(X_n)|t^4+8\varepsilon d^2|V(X_n)|\\
&=\left(\frac{1}{4}d^2-c_3\right)|V(X_n)|t^4+8\varepsilon d^2|V(X_n)|\\
&\leq \left(\frac{1}{4}d^2-\frac{1}{2}c_3\right)|V(X_n)|t^4\\
&=\left(1-\frac{2c_3}{d^2}\right)\left\|\frac{1}{2}dt^2\xi_n\right\|^2.
\end{aligned}
$$

For each R we have found an n_R, a $\frac{1}{3}R$-representation $\pi_{n_R} : B(\ell^2 X_{n_R}) \to B(\ell^2 X_{n_R})$, and a vector $\xi_{n_R} \in B(\ell^2 X_{n_R})$ satisfying $\left\|\pi_{n_R}(\Delta_X)\xi_{n_R} - \frac{1}{2}dt^2\xi_{n_R}\right\| \leq (1-\frac{2c_3}{d^2})^{\frac{1}{2}}\left\|\frac{1}{2}dt^2\xi_{n_R}\right\|$. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and define the ultraproduct $\mathcal{H} = \prod_n B(\ell^2 X_{n_R})/\mathcal{U}$. Let $\pi = \lim_{\mathcal{U}} \pi_{n_R}$ and $\xi = \lim_{\mathcal{U}} \frac{\xi_{n_R}}{\|\xi_{n_R}\|}$. Then π is a representation of $\mathbb{C}_{\mathrm{cs}}[X]$, and we have $\left\|\pi(\Delta_X)\xi - \frac{1}{2}dt^2\xi\right\| \leq (1-\frac{2c_3}{d^2})^{\frac{1}{2}}\left\|\frac{1}{2}dt^2\xi\right\|$. It follows that $\sigma(\pi(\Delta_X)) \cap [\frac{1}{2}dt^2(1-(1-\frac{2c_3}{d^2})^{\frac{1}{2}}), \frac{1}{2}dt^2(1+(1-\frac{2c_3}{d^2})^{\frac{1}{2}})] \neq \emptyset$. So $\sigma_{\max}(\Delta_X)$ contains a positive element that is at most dt^2. This is a contradiction with inequality (2). $\square$

Remark 3.2.6. Note that in the above proof, ε only depends on d and γ.

We have shown that each sequence of graphs with geometric property (T) has many small cycles. It follows that we can remove a large number of cycles of the graph, while still keeping the same coarse structure. In particular, the infimum in the definition of cost is not attained (see Definition 3.1.3).

Proposition 3.2.7. *Let $X = (X_n)$ be a sequence of graphs with degree at most d. Suppose there are $R, \varepsilon > 0$ such that $\dim Z_R(X_n) \geq \varepsilon|V(X_n)|$ for all n. Then $c(X) \leq \liminf_n \frac{|E(X_n)|}{|V(X_n)|} - \frac{\varepsilon}{d^{R-1}}$. In particular, if X has geometric property (T), the infimum $c(X) = \inf_Y \liminf_n \frac{|E(Y_n)|}{|V(X_n)|}$, over all sequences (Y_n) inducing the same coarse structure as X, is not attained.*

Proof. Consider the set of all subgraphs of X_n, on the same vertex set, without R-cycles. Let Y_n be a maximal element of this set. If two vertices x, y are connected in X_n, then either they are connected in Y_n, or adding the edge (x, y) to Y_n would create an R-cycle in Y_n. So $d_Y(x, y) \leq R-1$.

It follows that $d_Y(x, y) \leq (R-1)d_X(x, y)$ for any two vertices $x, y \in V(X_n)$. Since Y_n is a subgraph of X_n, we also have $d_Y(x, y) \geq d_X(x, y)$. This shows that X and Y induce the same coarse structure.

Since $\dim Z_R(X_n) \geq \varepsilon|V(X_n)|$, there are in particular at least $\varepsilon|V(X_n)|$ cycles of length at most R in X_n. Of all these cycles, at least one of its edges is not in Y_n. One edge can be contained in at most d^{R-1} cycles of length at most R. So $|E(Y_n)| \leq |E(X_n)| - \frac{\varepsilon}{d^{R-1}}|V(X_n)|$. This gives $\liminf_n \frac{|E(Y_n)|}{|V(X_n)|} \leq \liminf_n \frac{|E(X_n)|}{|V(X_n)|} - \frac{\varepsilon}{d^{R-1}}$. So $c(X) \leq \liminf_n \frac{|E(X_n)|}{|V(X_n)|} - \frac{\varepsilon}{d^{R-1}}$.

The last statement of the lemma follows from Theorem 3.A and the fact that if X and Y induce the same coarse structure, also Y has geometric property (T) (see [58, Theorem 4.1]). $\square$

3.3 Behaviour of geometric property (T) under small changes

A natural question in light of Theorem 3.A is the following: if a sequence of graphs (X_n) has geometric property (T), is it necessary that there is an R such that every vertex has a cycle in its R-neighbourhood? In this section we will see that the answer is no. We will do this by proving that if we change a sequence of graphs a small amount (see Definition 3.1.4) while keeping expansion, we also keep geometric property (T).

Theorem 3.B. Let $X = (X_n)$ and $Y = (Y_n)$ be approximately isomorphic sequences of finite connected graphs with uniformly bounded degree and number of vertices going to infinity. Suppose X has geometric property (T) and Y is an expander. Then Y has geometric property (T).

For the proof we need the following proposition.

Proposition 3.3.1. *Let $X = (X_n)$ be an expander sequence with maximum degree d, and let $h > 0$ such that $\sigma(\Delta_X) \subseteq \{0\} \cup [h, \infty)$. Let $\pi\colon \mathbb{C}_{\mathrm{cs}}[X] \to B(\mathcal{H})$ be a representation and suppose that $v \in \mathcal{H}$ is a unit vector with $\Delta_X v = \eta v$ for some $\eta > 0$.*

(i) Let $F_n \subseteq V(X_n)$ be such that $\lim_{n\to\infty} \frac{|F_n|}{|V(X_n)|} = 0$, and let $P_F \in \mathbb{C}_{\mathrm{cs}}[X]$ denote the projection on the vertices of the F_n. Then $\|P_F v\| \leq 2^{\frac{3}{4}} d^{\frac{1}{2}} h^{-\frac{1}{2}} \eta^{\frac{1}{4}}$.

(ii) Let $G = (G_n)$ be a sequence of subgraphs of X_n such that $\lim_{n\to\infty} \frac{|V(G_n)|}{|V(X_n)|} = 0$. Then $\langle \Delta_G v, v\rangle \leq 2^{\frac{3}{2}} d h^{-1} \eta^{\frac{3}{2}}$ and $\|\Delta_G v\| \leq 2^{\frac{3}{4}} d^{\frac{1}{2}} h^{-\frac{1}{2}} \eta^{\frac{5}{4}}$.

Proof. (i) We can assume that $\eta < h$. Let $\delta > 0$ and let N be an integer such that $|F_n|/|V(X_n)| < \delta$ for $n > N$. Let $P_{X_{\leq N}} \in \mathbb{C}_{\mathrm{cs}}[X]$ be the projection on the vertices of X_1 up to X_N. Then we have $\Delta_X P_{X_{\leq N}} v = \eta P_{X_{\leq N}} v$. Since η is not an eigenvalue of $\Delta_X P_{X_{\leq N}}$, we have $P_{X_{\leq N}} v = 0$. So $P_F v = P_{F_{>N}} v$, where $P_{F_{>N}}$ is the projection on the union $F_{>N} = \bigcup_{n>N} F_n$.

We can colour the edges of X in $2d$ colours such that no two intersecting edges have the same colour. For each colour i define the involution τ_i of X that sends each vertex in an edge with colour i to the other vertex of this edge, while fixing the other vertices. This defines an element in $\mathbb{C}_{\mathrm{cs}}[X]$ that we also denote by τ_i. Since $\tau_i^2 = 1$, we have $0 \leq 1 - \tau_i \leq 2$. This inequality holds in $C^*_{\max}(X)$. We have $\Delta_X = \sum_{i=1}^{2d} (1 - \tau_i)$: we can check this by computing the matrix coefficients. For any $x \in X$ we have $\sum_{i=1}^{2d} (1 - \tau_i)_{xx} = \#\{i \mid x \text{ has an edge with colour } i\} = \deg(x)$, and for different $x, y \in X$ we have $\sum_{i=1}^{2d} (1 - \tau_i)_{xy} = -\#\{i \mid (x, y) \text{ is an edge with colour } i\}$ which is -1 if (x, y) is an edge and 0 otherwise.

For $f \in \ell^\infty X$, denote the corresponding multiplication operator in $B(\ell^2 X)$ by M_f. Define the positive unital linear map $\varphi\colon \ell^\infty X \to \mathbb{C}$ by $\varphi(f) = \langle M_f v, v\rangle$. For each i we have $\langle (1-\tau_i)v, v\rangle \leq \eta$, so $\left\|(1 - \tau_i)v\right\|^2 = \langle (2 - 2\tau_i)v, v\rangle \leq 2\eta$. Then for $f \in \ell^\infty X$ we have

$$\begin{aligned}
(1 - \tau_i)\varphi(f) &= \varphi(f - f \circ \tau_i) \\
&= \langle M_f v, v\rangle - \langle M_{f\circ\tau_i} v, v\rangle \\
&= \langle M_f v, v\rangle - \langle M_f \tau_i v, \tau_i v\rangle \\
&= \langle M_f v, v - \tau_i v\rangle + \langle M_f (v - \tau_i v), \tau_i v\rangle.
\end{aligned}$$

We get $|(1 - \tau_i)\varphi(f)| \leq 2\|f\|_\infty \|v - \tau_i v\| \leq 2\sqrt{2\eta}\|f\|_\infty$, so $\left\|(1 - \tau_i)\varphi\right\| \leq 2\sqrt{2\eta}$.

So φ is almost invariant for the involutions τ_i. Note that φ is a unit vector in the Banach space $(\ell^\infty X)^*$. By the Goldstine theorem, φ is in the weak closure of the set $\mathcal{P}(X) \subseteq \ell^1 X$ of probability measures on X. We will use some functional analysis to show that we can approximate φ by a probability measure $\psi \in \mathcal{P}(X)$, in such a way that it will still be almost invariant for the involutions τ_i.

Let $\mathcal{B}$ be the real Banach space $\bigoplus_{i=1}^{2d} \ell^1(X, \mathbb{R})$ with norm $\|(\psi_i)\| = \max_i \|\psi_i\|_1$. Define the convex set $B \subseteq \mathcal{B}$ by

$$B = \{((1-\tau_i)\psi) \mid \psi \in \mathcal{P}(X), |\psi(\mathbb{1}_{F_{>N}}) - \varphi(\mathbb{1}_{F_{>N}})| < \delta\}.$$

There is a net $\psi_\lambda \in \mathcal{P}(X)$ converging weakly to φ. For large λ we have $|\psi_\lambda(\mathbb{1}_{F_{>N}}) - \varphi(\mathbb{1}_{F_{>N}})| < \delta$. Now $(1-\tau_i)\psi_\lambda$ converges weakly to $(1-\tau_i)\varphi \in (\ell^\infty X)^*$ for all i. So $((1-\tau_i)\varphi) \in \mathcal{B}^{**}$ is in the weak closure of B.

Let A be the open ball $\{a \in \mathcal{B} \mid \|a\| < 2\sqrt{2\eta} + \delta\}$. Suppose A and B are disjoint. By the Hahn-Banach separation theorem, there is $f \in \mathcal{B}^*$ and a positive real number s with $f(a) < s \leq f(b)$ for $a \in A$ and $b \in B$. Now $((1-\tau_i)\varphi) \in B^{**}$ is in the weak closure of B, so $f(((1-\tau_i)\varphi)) \geq s$. On the other hand, $\|((1-\tau_i)\varphi)\| \leq 2\sqrt{2\eta}$, so $((1-\tau_i)\varphi)$ is in the weak closure of $\frac{2\sqrt{2\eta}}{2\sqrt{2\eta}+\delta} A$, showing that $f(((1-\tau_i)\varphi)) \leq \frac{2\sqrt{2\eta}}{2\sqrt{2\eta}+\delta} s$, giving a contradiction.

Therefore $A \cap B \neq \emptyset$, so there is $\psi \in \mathcal{P}(X)$ with $|\psi(\mathbb{1}_{F_{>N}}) - \varphi(\mathbb{1}_{F_{>N}})| < \delta$ and $\|(1-\tau_i)\psi\| < 2\sqrt{2\eta} + \delta$ for all i. Let $\xi = \psi^{\frac{1}{2}} \in \ell^2 X$. Then for all i, we have

$$\begin{aligned}
\langle (1-\tau_i)\xi, \xi \rangle &= \sum_{x \in X} (\xi(x) - \xi(\tau_i x))\xi(x) \\
&= \frac{1}{2} \sum_{x \in X} \left(\xi(x) - \xi(\tau_i x)\right)^2 \\
&\leq \frac{1}{2} \sum_{x \in X} |\xi(x)^2 - \xi(\tau_i x)^2| \\
&= \frac{1}{2} \|(1-\tau_i)\psi\|_1 \\
&\leq \sqrt{2\eta} + \frac{1}{2}\delta.
\end{aligned}$$

Summing over all i gives $\langle \Delta_X \xi, \xi \rangle \leq 2d\sqrt{2\eta} + d\delta$. Let ξ_c be the projection of ξ on the locally constant functions in $\ell^2 X$. We get $\langle \Delta_X(\xi - \xi_c), \xi - \xi_c \rangle = \langle \Delta_X \xi, \xi \rangle \leq 2d\sqrt{2\eta} + d\delta$, but also $\langle \Delta_X(\xi - \xi_c), \xi - \xi_c \rangle \geq h\|\xi - \xi_c\|_2^2$. So $\|\xi - \xi_c\|_2^2 \leq \frac{1}{h}(2d\sqrt{2\eta} + d\delta)$.

Since $\frac{|F_n|}{V(X_n)} < \delta$ for $n > N$, we have $\left\|\xi_{c|F_{>N}}\right\| < \sqrt{\delta}$. We get

$$\begin{aligned}
\psi(\mathbb{1}_{F_{>N}}) &= \sum_{x \in F_{>N}} \xi(x)^2 \\
&= \sum_{x \in F_{>N}} (\xi - \xi_c)(x)^2 + 2 \sum_{x \in F_{>N}} \xi_c(x)(\xi - \xi_c)(x) + \sum_{x \in F_{>N}} \xi_c(x)^2 \\
&\leq \|\xi - \xi_c\|_2^2 + 2\left\|\xi_{c|F_{>N}}\right\|_2 \cdot \|\xi - \xi_c\|_2 + \left\|\xi_{c|F_{>N}}\right\|_2^2
\end{aligned}$$

$$\leq \frac{1}{h}(2d\sqrt{2\eta} + d\delta) + 2\sqrt{\delta} + \delta.$$

Finally, we have

$$\|P_F v\|^2 = \|P_{F>N} v\|^2 = \langle M_{\mathbb{1}_{F>N}} v, v\rangle = \varphi(\mathbb{1}_{F>N}) \leq \psi(\mathbb{1}_{F>N}) + \delta \leq \frac{1}{h}(2d\sqrt{2\eta} + d\delta) + 2\sqrt{\delta} + 2\delta.$$

Letting $\delta \to 0$ gives the desired conclusion.

(ii) We recursively define $F_0 = \bigcup_n V(G_n)$ and

$$F_{k+1} = \{x \in V(X) \setminus F_{\leq k} \mid x \text{ is adjacent to a vertex in } F_k\}$$

for $k \geq 1$. Here we write $F_{\leq k}$ for $F_0 \cup \ldots \cup F_k$. Let $\delta > 0$. Since there are infinitely many F_k and they are all disjoint, there is $k \geq 1$ with $\left\|P_{F_{k-1} \cup F_k} v\right\| \leq \delta$. Since the graphs have uniformly bounded degree, we have $\lim_{n\to\infty} \frac{|F_{\leq k-1}|}{|V(X_n)|} = 0$. By part (i), we have $\left\|P_{F_{\leq k-1}} v\right\| \leq 2^{\frac{3}{4}} d^{\frac{1}{2}} h^{-\frac{1}{2}} \eta^{\frac{1}{4}}$. Denote by $\Delta_{F_{\leq k}}$ the Laplacian operator of the induced subgraph with vertex set $F_{\leq k}$. Then we have

$$P_{F_{\leq k-1}} \Delta_X = P_{F_{\leq k-1}} \Delta_{F_{\leq k}} = \Delta_{F_{\leq k}} - P_{F_k} \Delta_{F_{\leq k}} = \Delta_{F_{\leq k}} - P_{F_k} \Delta_{F_{\leq k}} P_{F_{k-1} \cup F_k}.$$

It follows that

$$\begin{aligned}
\langle \Delta_G v, v\rangle &\leq \langle \Delta_{F_{\leq k}} v, v\rangle \\
&= \langle P_{F_{\leq k-1}} \Delta_X v, v\rangle + \langle P_{F_k} \Delta_{F_{\leq k}} P_{F_{k-1}\cup F_k} v, v\rangle \\
&\leq \eta \langle P_{F_{\leq k-1}} v, v\rangle + \left\|P_{F_k} \Delta_{F_{\leq k}}\right\| \cdot \left\|P_{F_{k-1}\cup F_k} v\right\| \cdot \|v\| \\
&\leq 2^{\frac{3}{2}} d h^{-1} \eta^{\frac{3}{2}} + 2d\delta.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|\Delta_G v\| &\leq \left\|F_{\leq k} v\right\| \\
&\leq \left\|P_{F_{\leq k-1}} \Delta_X v\right\| + \left\|P_{F_k} \Delta_{F_{\leq k}} P_{F_{k-1}\cup F_k} v\right\| \\
&\leq 2^{\frac{3}{4}} d^{\frac{1}{2}} h^{-\frac{1}{2}} \eta^{\frac{5}{4}} + 2d\delta.
\end{aligned}$$

Since these inequalities hold for all $\delta > 0$, the conclusion follows.

$\square$

Proof of Theorem 3.B. Since X and Y are approximately isomorphic, we can identify the isomorphic subgraphs and assume there are induced subgraphs $Z_n \subseteq X_n \cap Y_n$ with $\lim_{n\to\infty} \frac{|V(Z_n)|}{|V(X_n)|} = \lim_{n\to\infty} \frac{|V(Z_n)|}{|V(Y_n)|} = 1$. Let d be the maximum degree in $X \cup Y$, let $\gamma > 0$ with $\sigma_{\max}(\Delta_X) \subseteq \{0\} \cup [\gamma, \infty)$ and let $h > 0$ with $\sigma(\Delta_Y) \subseteq \{0\} \cup [h, \infty)$. Suppose that Y does not have geometric property (T). Then the maximal spectrum of Δ_Y contains arbitrarily small positive numbers. Let $0 < \eta < h$ be in the maximal spectrum of Δ_Y. Then there is a representation $\rho\colon \mathbb{C}_{\mathrm{cs}}[Y] \to B(\mathcal{H})$ and a unit vector $v \in \mathcal{H}$ with $\Delta_Y v = \eta v$.

We will first apply Proposition 3.3.1 to bound v on some small subsets of Y. Then we will construct a representation of $\mathbb{C}_{\mathrm{cs}}[X]$ containing a vector $1 \otimes v$, that we will show is almost constant. Since X has geometric property (T), it follows that $1 \otimes v$ is close to some constant vector. This will give a contradiction because v is perpendicular to all constant vectors.

We have

$$\langle \Delta_{Y \setminus Z} v, v \rangle \leq 2^{\frac{3}{2}} d h^{-1} \eta^{\frac{3}{2}}$$

and

$$\left\| \Delta_{Y \setminus Z} v \right\| \leq 2^{\frac{3}{4}} d^{\frac{1}{2}} h^{-\frac{1}{2}} \eta^{\frac{5}{4}}$$

by Proposition 3.3.1(ii).

Let $F = \{z \in Z \mid z \text{ adjacent to some } x \in X \setminus Z\}$. Since the degree of the vertices of Y is uniformly bounded, we have $\lim_{n\to\infty} \frac{|F \cap Y_n|}{|Y_n|} = 0$. By Proposition 3.3.1(i), we have $\|P_F v\| \leq 2^{\frac{3}{4}} d^{\frac{1}{2}} h^{-\frac{1}{2}} \eta^{\frac{1}{4}}$.

Now we construct a representation of $\mathbb{C}_{\mathrm{cs}}[X]$. Consider the map $E\colon \mathbb{C}_{\mathrm{cs}}[X] \to \mathbb{C}_{\mathrm{cs}}[Z]$ given by $E(T) = P_Z T P_Z$. This is a conditional expectation, meaning that for $T \in \mathbb{C}_{\mathrm{cs}}[X]$ and $S \in \mathbb{C}_{\mathrm{cs}}[Z]$, we have $E(TS) = E(T)S$ and $E(ST) = SE(T)$. We can construct the tensor product $\mathcal{H}' = \mathbb{C}_{\mathrm{cs}}[X] \otimes_{\mathbb{C}_{\mathrm{cs}}[Z]} \mathcal{H}$, as in [42, Theorem 1.8]. We repeat the construction here. First we consider the algebraic tensor product $\mathbb{C}_{\mathrm{cs}}[X] \odot \mathcal{H}$. We equip this with a conjugate symmetric form given on simple tensors by

$$\langle T_1 \odot v_1, T_2 \odot v_2 \rangle = \langle E(T_2^* T_1) v_1, v_2 \rangle.$$

It can be shown that this is positive semi-definite (see [42, Lemma 1.7] and its proof). Define the semi-norm $\|w\| = \langle w, w \rangle^{\frac{1}{2}}$ for $w \in \mathbb{C}_{\mathrm{cs}}[X] \odot \mathcal{H}$. Let $\mathcal{H}' = \mathbb{C}_{\mathrm{cs}}[X] \otimes_{\mathbb{C}_{\mathrm{cs}}[Z]} \mathcal{H}$ be the Hilbert space we get by taking the quotient with respect to the kernel of $\|\cdot\|$ and then taking the completion. Let $T_1 \otimes v_1$ denote the image of $T_1 \odot v_1$ in $\mathcal{H}'$. It is easy to see that for $S \in \mathbb{C}_{\mathrm{cs}}[Z]$, we have $T_1 S \otimes v_1 = T_1 \otimes S v_1$. We now have a representation $\pi\colon \mathbb{C}_{\mathrm{cs}}[X] \to B(\mathcal{H}')$, given on simple tensors by $\pi(T_1)(T_2 \otimes v_1) = T_1 T_2 \otimes v_1$.

Consider the unit vector $1 \otimes v \in \mathcal{H}'$. We have

$$\begin{aligned}
\langle \Delta_X (1 \otimes v), 1 \otimes v \rangle &= \langle \Delta_Z v, v \rangle + \langle E(\Delta_{X \setminus Z}) v, v \rangle \\
&= \langle \Delta_Z v, v \rangle + \langle E(\Delta_{X \setminus Z}) P_F v, P_F v \rangle \\
&\leq \langle \Delta_Y v, v \rangle + \left\| E(\Delta_{X \setminus Z}) \right\| \cdot \|P_F v\|^2 \\
&\leq \eta + 2d \cdot 2^{\frac{3}{2}} d h^{-1} \eta^{\frac{1}{2}} \\
&\leq 8 d^2 h^{-1} \eta^{\frac{1}{2}},
\end{aligned}$$

provided η is small enough. Let $w \in \mathcal{H}'_c$ be the projection of $1 \otimes v$ on the space of constant vectors $\mathcal{H}'_c = \ker(\rho(\Delta_X))$. Then we have

$$\langle \Delta_X (1 \otimes v), 1 \otimes v \rangle = \langle \Delta_X (1 \otimes v - w), 1 \otimes v - w \rangle \geq \gamma \|1 \otimes v - w\|^2 .$$

Combining these inequalities, we get

$$\|1 \otimes v - w\|^2 \leq 8 d^2 h^{-1} \gamma^{-1} \eta^{\frac{1}{2}} .$$

Finally, we have

$$\eta = \langle \Delta_Y v, v \rangle$$

$$
\begin{aligned}
&= \langle \Delta_Z v, v\rangle + \langle \Delta_{Y\setminus Z} v, v\rangle \\
&= \langle \Delta_Z(1\otimes v), w\rangle + \langle \Delta_Z(1\otimes v), 1\otimes v - w\rangle + \langle \Delta_{Y\setminus Z} v, v\rangle \\
&= \langle \Delta_Z(1\otimes v), 1\otimes v - w\rangle + \langle \Delta_{Y\setminus Z} v, v\rangle \\
&= \langle \eta(1\otimes v), 1\otimes v - w\rangle - \langle 1\otimes \Delta_{Y\setminus Z} v, 1\otimes v - w\rangle + \langle \Delta_{Y\setminus Z} v, v\rangle \\
&\leq \eta\|1\otimes v - w\|^2 + \left\|\Delta_{Y\setminus Z} v\right\| \cdot \|1\otimes v - w\| + 2^{\frac{3}{2}} d h^{-1} \eta^{\frac{3}{2}} \\
&\leq \left(8d^2h^{-1}\gamma^{-1} + 2^{\frac{9}{4}} d^{\frac{3}{2}} h^{-1}\gamma^{-\frac{1}{2}} + 2^{\frac{3}{2}} d h^{-1}\right)\eta^{\frac{3}{2}}.
\end{aligned}
$$

This gives a contradiction if η is small enough. Hence Y must have geometric property (T). □

Using the theorem, we can construct a sequence of graphs with geometric property (T) such that the graphs locally have arbitrarily large girth.

Corollary 3.3.2. *There is a sequence of graphs (X_n) with uniformly bounded degree and number of vertices converging to infinity, satisfying geometric property (T), with a designated vertex $p_n \in V(X_n)$, such that the n-ball around p_n does not contain any cycles.*

Proof. We start with a sequence of finite connected graphs (Y_n) with geometric property (T) and maximal degree d. We will construct the graphs X_n by attaching new trees to each Y_n.

Let (R_n) be an unbounded sequence of integers with $\lim_{n\to\infty} \frac{2^{R_n}}{|V(Y_n)|} = 0$. Let (T_n, p_n) be a rooted tree of depth R_n such that each vertex except for the leaves has degree 3. Connect each leaf of the tree to a different vertex in Y_n. We call the new graph X_n, and we show that (X_n) is still an expander sequence. Note that the maximal degree of X_n equals $d+1$. For a subset $A \subseteq V(X_n)$, denote by $\partial_{\text{out}} A$ the outer vertex boundary, that is the set $\{x \in V(X_n) \mid x \notin A,\ x \text{ adjacent to a vertex in } A\}$. Since (Y_n) is an expander sequence, there is $h > 0$ such that for all n and all $B \subseteq V(Y_n)$ with $|B| \leq \frac{2}{3}|Y_n|$, we have $|\partial_{\text{out}} B| \geq h|B|$. Now let $C \subseteq V(X_n)$ with $|C| \leq \frac{1}{2}|V(X_n)|$. Write $A = C \cap V(T_n)$ and $B = C \cap V(Y_n)$. Note that $|B| \leq \frac{2}{3}|V(Y_n)|$ (provided n is large enough). We have $|\partial_{\text{out}} A| \geq \frac{1}{2}|A|$, and all vertices in $\partial_{\text{out}} A$ are also in $\partial_{\text{out}} C$ unless they are in B, so $|\partial_{\text{out}} C| \geq \frac{1}{2}|A| - |B|$. We also have $|\partial_{\text{out}} C| \geq |\partial_{\text{out}} B \cap V(Y_n)| \geq h|B|$. Taking a convex combination, we conclude that

$$
|\partial_{\text{out}} C| \geq \left(1 + \frac{3}{2h}\right)^{-1} \left(\frac{1}{2}|A| - |B| + \frac{3}{2h} h|B|\right) = \frac{h}{2h+3}|C|.
$$

Hence, (X_n) is an expander sequence.

By the condition on R_n, we see that (X_n) approximates (Y_n). By Theorem 3.B, the sequence (X_n) has geometric property (T). The R_n-neighbourhood of p_n is the tree T_n, so it does not contain any cycles. After taking a subsequence and renumbering the graphs, we get the sequence of graphs we wanted. □

4 Geometric property (T) for non-discrete spaces

This chapter is based on the article [54]. Large parts of it are reproduced verbatim.

Chapter summary

Geometric property (T) was defined by Willett and Yu, first for sequences of graphs and later for more general discrete spaces. Increasing sequences of graphs with geometric property (T) are expanders, and they are examples of coarse spaces for which the maximal coarse Baum-Connes assembly map fails to be surjective. Here, we give a broader definition of bounded geometry for coarse spaces, which includes non-discrete spaces. We define a generalisation of geometric property (T) for this class of spaces and show that it is a coarse invariant. Additionally, we characterise it in terms of spectral properties of Laplacians. We investigate geometric property (T) for manifolds and warped systems.

4.1 Introduction and main results

In their paper [57], Willett and Yu introduce *geometric property (T)* for sequences of graphs, in order to provide examples of coarse spaces for which the maximal coarse Baum-Connes assembly map fails to be surjective. The concept is studied further by the same authors in [58]. There, geometric property (T) is defined for coarse spaces that are uniformly locally finite, monogenic, and countable. Such a space has geometric property (T) if each unitary representation of the translation algebra $\mathbb{C}_u[X]$ (we call it $\mathbb{C}_{\mathrm{cs}}[X]$, see Definition 4.5.3) that has almost invariant vectors, has a non-zero invariant vector. For increasing sequences of finite graphs, this is a strictly stronger property than being an expander. In fact, in this case it is equivalent to the graph Laplacian having spectral gap in the maximal completion of $\mathbb{C}_u[X]$ (see [58, Proposition 5.2]). A sequence of finite graphs with geometric property (T) can never have large girth (see [57, Corollary 7.5]).

The aim of this chapter is to generalise geometric property (T) to spaces satisfying a broader notion of *bounded geometry*, see Definition 4.6.7. This broader notion of bounded geometry is explained in Section 4.3. Spaces with bounded geometry include sequences of graphs with bounded degree, manifolds whose Ricci curvature and injectivity radius are uniformly bounded from below, and warped systems coming from an action of a finitely generated group on a compact manifold. We do not assume monogenicity of the spaces. However, we do assume that the spaces are covered by a countable number of "bounded" sets.

In [58], *translation operators* are used extensively to study geometric property (T) for discrete spaces. For non-discrete spaces, these operators do not behave as nicely. Therefore we introduce the concept of "operators in blocks", which are operators with a convenient decomposition (see Definition 4.5.5).

Any coarse space with bounded geometry can be equipped with a measure μ that is *uniformly bounded* and for which a *gordo* set exists (see Definition 4.3.6 and Proposition 4.3.7). We will firstly define geometric property (T) for a coarse space X equipped with such a measure μ. However, Theorem 4.A will show that it is in fact independent of the chosen measure μ.

We will consider the Hilbert space $L^2(X,\mu)$. An operator T in $B(L^2(X,\mu))$ has *controlled support* if the support of a function is not changed much by T (see Definition 4.5.1). Let $\mathbb{C}_{\mathrm{cs}}[X] \subseteq B(L^2(X,\mu))$ be the algebra of operators with controlled support. A representation of this algebra is a unital $*$-homomorphism $\rho\colon \mathbb{C}_{\mathrm{cs}}[X] \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. For such a representation, the subspace of *constant vectors* $\mathcal{H}_c$ is defined (see Definition 4.6.6). The coarse space X has geometric property (T) if every unit vector $v \in \mathcal{H}_c^\perp$ is "moved enough" by a suitable operator in $\mathbb{C}_{\mathrm{cs}}[X]$ (see Definition 4.6.7).

An important notion in coarse geometry is that of *coarse invariance* (see Section 4.2). In [58], it was shown that geometric property (T) is a coarse invariant (see [58, Theorem 4.1]). In this chapter we show that our generalisation of geometric property (T) is still a coarse invariant. This gives evidence that we have the "right" definition of property (T).

Theorem 4.A. Suppose (X, μ) and (X', μ') are coarsely equivalent spaces with bounded geometry, equipped with uniformly bounded measures for which gordo sets exist. Then (X, μ) has property (T) if and only if (X', μ') has property (T). In particular, whether X has property (T) is independent of the chosen uniformly bounded measure μ for which a gordo set exists.

Analogous to [58], we establish a complete characterisation of geometric property (T) in terms of non-amenability for connected coarse spaces of bounded geometry. In fact, for unbounded connected spaces, geometric property (T) is the complete converse of amenability for coarse spaces. This generalises [58, Corollary 6.1]. The notion of amenability for coarse spaces with bounded geometry is introduced in Lemma 4.9.1.

Theorem 4.B. Let X be a connected coarse space and let μ be a uniformly bounded measure for which a gordo set exists. If X is bounded, then it is amenable and has geometric property (T). If X is unbounded, it has geometric property (T) if and only if it is not amenable.

Towards the end of this chapter we investigate geometric property (T) for Riemannian manifolds. Let M be a Riemannian manifold or a countable disjoint union of Riemannian manifolds with the same dimension. The Riemannian metric defines a coarse structure on M. If the Ricci curvature and the injectivity radius of M are uniformly bounded from below, then M has bounded geometry (see Section 4.10). In this case the manifold Laplacian Δ_M is an element of the algebra $C^*_{\max}(X)$, and the manifold M has geometric property (T) if and only if Δ_M has spectral gap in this algebra.

Theorem 4.C. Let (M, g) be a Riemannian manifold of bounded geometry equipped with the coarse structure coming from the geodesic metric. Then M has geometric property (T) if and only if the Laplacian Δ_M has spectral gap in $C^*_{\max}(M)$.

In the last section we discuss warped systems. A group Γ acting by homeomorphisms on a Riemannian manifold M gives rise to a coarse spaces called a *warped system*, see Section 4.11. In Section 4.11 we will show that warped systems have bounded geometry if the manifold M is compact. One might expect that the warped system has geometric property (T) if and only if the group has property (T) and the action is ergodic. This remains an open question, though some partial results are given.

4.2 Notation on coarse spaces

Coarse spaces are spaces with a large-scale geometric structure. See Section 2.1 for the definition of coarse spaces and coarse equivalences. In this chapter we will use the following notation:

Definition 4.2.1. Let X be a coarse space and $E \subseteq X \times X$ a controlled set.

(i) The controlled set E is called *symmetric* if $E^{-1} = E$.

(ii) For any $x \in X$, we write $E_x = \{y \in X \mid (y, x) \in E\}$.

(iii) For any $U \subseteq X$, we write $E_U = \bigcup_{x \in U} E_x$.

(iv) A subset $U \subseteq X$ is called *bounded* if $U \times U$ is a controlled set.

(v) A subset $U \subseteq X$ is called *E-bounded* if $U \times U \subseteq E$.

(vi) We write $E^{\circ n}$ for the n-fold composition $E \circ E \circ \cdots \circ E$.

(vii) The set E *generates* the coarse structure if for any controlled set F, there is an integer n such that $F \subseteq E^{\circ n}$.

For later use we give some basic properties using these definitions.

Lemma 4.2.2. *Let X be a coarse space.*

(i) Each controlled set is contained in a symmetric controlled set.

(ii) A subset $U \subseteq X$ is bounded if and only if it is E-bounded for some controlled set E.

(iii) If E is a controlled set and $x \in X$ then E_x is $E \circ E^{-1}$-bounded (though it is not necessarily E-bounded).

(iv) If U is an E-bounded set, it is contained in E_x for any $x \in U$.

Proof. These follow directly from the definitions. □

Most of the time, we will only consider symmetric controlled sets.

4.3 Bounded geometry

Consider a coarse space X. It is *uniformly locally finite* if for every controlled set $E \subseteq X \times X$ there is an integer N such that $\#E_x \leq N$ for all $x \in X$. We will define geometric property (T) for a wider class of spaces. We still need a concept of bounded geometry, which we define below.

Definition 4.3.1. Let X be a coarse space. A controlled set $F \subseteq X \times X$ is called *covering* if it is symmetric and for every controlled E there is an N such that each E_x can be covered by at most N sets that are F-bounded.

Lemma 4.3.2. *Let X be a coarse space and $F \subseteq X \times X$ a symmetric controlled set. The following are equivalent:*

(i) The controlled set F is covering.

(ii) For each controlled set E there is a constant N such that each E-bounded set U can be covered by at most N sets that are F-bounded.

Proof. Suppose F is covering and let E be a controlled set. Let N be such that each E_x can be covered by at most N sets that are F-bounded. Let U be a non-empty E-bounded set. For any $x \in U$ we have $U \subseteq E_x$, so U is also covered by at most N sets that are F-bounded.

Conversely, suppose that (ii) is true and let E be a controlled set. There is a constant N such that each $E \circ E^{-1}$-bounded set can be covered by at most N sets that are F-bounded. Then each E_x can also be covered by at most N sets that are F-bounded. □

Definition 4.3.3. A space X has bounded geometry if there is a controlled covering set $F \subseteq X \times X$.

The definition above is easily seen to be equivalent to the definition given by Roe in [44, Definition 3.8].

Example 4.3.4. Consider a metric space X with the corresponding coarse structure. If there exists a controlled covering set, we can enlarge it to E_R for some $R > 0$. Therefore, X has bounded geometry if and only if E_R is covering for some $R > 0$.

In fact, for many examples of metric spaces, E_R will be covering for any $R > 0$.

Example 4.3.5. A uniformly locally finite space certainly has bounded geometry in this definition, as we can take F to be the diagonal.

A different way to characterise bounded geometry is by the existence of a measure on X satisfying certain properties. We will study measures on any sigma algebra on the set X, with as only condition, that singletons must be measurable. A controlled set $E \subseteq X$ acts on the set of subsets of X, by sending U to E_U. Accordingly, given a measure on X, we say that a controlled set E is *measurable* if E_U is measurable for all measurable $U \subseteq X$. In particular, E_x is then measurable. Note that $(E \circ F)_U = E_{F_U}$, therefore the composition of measurable controlled sets is again measurable.

Definition 4.3.6. Let $(X, \mathcal{E})$ be a coarse space, and let μ be a measure on X, for any sigma algebra that contains all singletons.

(i) The measure is called *$\mathcal{E}$-uniformly bounded* or simply *uniformly bounded* if for each controlled set E there is a constant $C > 0$ such that each E-bounded measurable set U has measure at most C.

(ii) A symmetric controlled set E is called *μ-gordo* or simply *gordo* if it is measurable and $\mu(E_x)$ is bounded away from zero independently of x, for all $x \in X$.

Proposition 4.3.7. *Let X be a coarse space. The following are equivalent:*

(i) The space X has bounded geometry.

(ii) There is a uniformly bounded measure μ on X and a gordo set E for μ.

(iii) There is a coarsely dense subspace $Y \subseteq X$ that is uniformly locally finite.

Proof. Suppose X has bounded geometry and $F \subseteq X \times X$ is a controlled covering set. By Zorn's Lemma there is a maximal subset $Y \subseteq X$ satisfying $(y, y') \notin F$ for all $y \neq y' \in Y$. By maximality we know that for each $x \in X$ there exists $y \in Y$ such that $(x, y) \in F$, so Y is coarsely dense in X. To see that Y is uniformly locally finite, consider a controlled set $E \subseteq X \times X$. There exists N such that every E_y can be covered by F-bounded sets $U_1, \ldots, U_N$. Since each F-bounded set can contain at most 1 element of Y, we see that $\#(E_y \cap Y) \leq N$, so Y has bounded geometry, proving (i) $\Longrightarrow$ (iii).

Suppose $Y \subseteq X$ is a coarsely dense, uniformly locally finite subspace and let μ be the counting measure of Y. Any controlled set E on X restricts to a controlled set on Y. Since Y is uniformly locally finite any E-bounded set can contain a bounded number of points of Y. This shows that μ is uniformly bounded. Now let $E \subseteq X \times X$ be a symmetric controlled set such that for all $x \in X$ there is $y \in Y$ with $(x, y) \in E$. Then each E_x has measure at least 1, so E is gordo, showing (iii) $\Longrightarrow$ (ii).

Finally, let μ be a uniformly bounded measure on X and let F be a gordo set for μ. We will show that $F^{\circ 4}$ is a covering set. There is $\varepsilon > 0$ such that $\mu(F_x) \geq \varepsilon$ for all x. Let E be a symmetric controlled set. Since μ is uniformly bounded, there is a constant C such that $\mu(V) \leq C$ for all $F \circ E \circ F$-bounded measurable V. Now let $U \subseteq X$ be E-bounded. Then F_U is $F \circ E \circ F$-bounded, so all its measurable subsets have measure at most C. Let $Y \subseteq U$ be maximal such that the F_y are pairwise disjoint when the y range over Y. Since the F_y are all contained in F_U and have measure at least ε we have $\#Y \leq N$, with $N = \frac{C}{\varepsilon}$. Since Y is maximal we know that for all $x \in U$ there is a $y \in Y$ with $F_x \cap F_y \neq \emptyset$, so $(x, y) \in F \circ F$. This implies that the sets $(F \circ F)_y, y \in Y$ cover U. These sets are $F^{\circ 4}$-bounded. By Lemma 4.3.2 we see that $F^{\circ 4}$ is covering, showing (ii) $\implies$ (i). $\square$

4.4 Blocking collections

From now on, let X be a coarse space of bounded geometry that can be covered by a countable number of bounded sets. Let μ be a uniformly bounded measure on X for which a gordo set exists. This measure exists by Proposition 4.3.7; in applications, there is often already a measure given that satisfies the condition above. For example, if X is a metric space and μ is a Borel measure, then μ satisfies the conditions if there is a radius $R > 0$ such that R-balls are bounded from below in measure, and for all radii $R > 0$ the R-balls are bounded from above in measure.

The definition of geometric property (T) uses the measure μ, but we will show later that it does not depend on it (see Theorem 4.A).

We will start by defining the notion of a blocking collection of subsets of X and proving some elementary results about them.

Definition 4.4.1. Let E be a controlled set and let $\varepsilon > 0$. A collection (A_i) of measurable subsets of X is called *(E, ε)-blocking* if $\bigcup_i A_i \times A_i \subseteq E$, the A_i are pairwise disjoint and $\mu(A_i) \geq \varepsilon$ for all i. A collection (A_i) of measurable subsets of X is called *blocking* if it is (E, ε)-blocking for some E, ε.

The conditions on the measure μ ensure that there are sufficiently large blocking collections:

Lemma 4.4.2. *There is a blocking collection (A_i) with union X.*

Proof. Let $E \subseteq X \times X$ be a gordo set for μ. By Zorn's Lemma, there is a maximal subset $I \subseteq X$ such that all the E_i are pairwise disjoint when i ranges over I. Then for all $x \in X$ there exists an $i \in I$ such that $E_x \cap E_i \neq \emptyset$, so $(x, i) \in E \circ E$. So the $(E \circ E)_i$ cover X, while the E_i are pairwise disjoint. Hence, we can find measurable A_i with $E_i \subseteq A_i \subseteq (E \circ E)_i$ such that $X = \bigsqcup_i A_i$. To construct the A_i explicitly, choose a well-ordering on I and define

$$A_i = E_i \cup \left((E \circ E)_i \setminus \left(\bigcup_{i' \neq i} E_{i'} \cup \bigcup_{i' < i} (E \circ E)_{i'} \right) \right).$$

Now the measure of the A_i is bounded from below because E is gordo, and the A_i are $E^{\circ 4}$-bounded, showing that the A_i form a blocking collection. $\square$

Definition 4.4.3. Let $E \subseteq X \times X$. We say that E is a *controlled set in blocks* if E is of the form $\bigsqcup_i A_i \times A_i$, where the (A_i) form a blocking collection.

Note that $\bigsqcup_i A_i \times A_i$ is always a controlled set if the (A_i) form a blocking collection.

Lemma 4.4.4. *Let $E \subseteq X \times X$ be a controlled set. Then E is contained in a finite union of controlled sets in blocks.*

Proof. Assume without loss of generality that E is symmetric. By Lemma 4.4.2, there is a blocking collection (A_i) with union X. Let $F = \bigsqcup_i A_i \times A_i$ which is a controlled set. Define a graph structure on the index set I by connecting i and j if $E_{A_i} \cap E_{A_j} \neq \emptyset$.

If i and j are connected by an edge, then $(E \circ E)_{A_i} \cap A_j \neq \emptyset$, and since A_j is F-bounded it follows that $A_j \subseteq (F \circ E \circ E)_{A_i}$. Since μ is uniformly bounded and the A_i are F-bounded, we know that the measure of $(F \circ E \circ E)_{A_i}$ is bounded from above, say by M, where M does not depend on i. Moreover, the measure of A_j is bounded from below uniformly in j, say by $\varepsilon > 0$. Since the A_j are also pairwise disjoint, the degree of any vertex i in the graph can be at most $\frac{M}{\varepsilon}$. Then we can colour the graph in N colours, for some integer N, such that no two adjacent vertices have the same colour. Now for $1 \leq k \leq N$ define

$$\mathcal{C}_k = \{E_{A_i} \mid i \text{ is coloured with the } k\text{-th colour}\}.$$

Then because of the way we coloured the graph, the sets in $\mathcal{C}_k$ are disjoint. Moreover they are $E \circ F \circ E$-bounded and bounded from below in measure, so $\mathcal{C}_k$ is a blocking collection for each k. For $1 \leq k \leq N$ we now define

$$E_k = \bigsqcup \{E_{A_i} \times E_{A_i} \mid i \text{ is coloured with the } k\text{-th colour}\}.$$

These are controlled sets in blocks. For all $(x, y) \in E$, there is an i with $y \in A_i$. Let k be the colour of i. Then $(x, y) \in E_{A_i} \times E_{A_i} \subseteq E_k$. So E is contained in $\bigsqcup_k E_k$. $\square$

4.5 The algebra of a coarse space

Analogous to the discrete setting, we now define the algebra of operators with controlled support. We will use the Hilbert space $L^2 X = L^2(X, \mu)$.

Definition 4.5.1. Let $E \subseteq X \times X$ be a measurable controlled set and $T \in B(L^2 X)$. We say that T is *supported on* E if for all measurable $U \subseteq X$, all $\xi \in L^2 X$ supported on U and all $\eta \in L^2 X$ we have

$$\langle \xi, T\eta \rangle = \langle \xi, T\eta|_{E_U} \rangle.$$

We say that T has *controlled support* if it is supported on some measurable controlled set.

We can give some equivalent definitions for this:

Lemma 4.5.2. *Let E be a measurable symmetric controlled set and $T \in B(L^2 X)$. The following are equivalent:*

(i) The operator T is supported on E.

(ii) For all measurable $U \subseteq X$, all $\xi \in L^2 X$ and all $\eta \in L^2 X$ supported on U we have

$$\langle \xi, T\eta \rangle = \langle \xi|_{E_U}, T\eta \rangle.$$

(iii) The operator T^* is supported on E.

(iv) For all measurable $U \subseteq X$ and all $\eta \in L^2X$ supported on U the function $T\eta$ is supported on E_U.

Proof. To prove (i) $\implies$ (ii), we have

$$\langle \xi, T\eta\rangle = \langle \xi|_{E_U}, T\eta\rangle + \langle \xi|_{X\setminus E_U}, T\eta\rangle = \langle \xi|_{E_U}, T\eta\rangle + \langle \xi|_{X\setminus E_U}, T\eta|_{E_{X\setminus E_U}}\rangle = \langle \xi|_{E_U}, T\eta\rangle.$$

The other direction is similar, and the equivalences (ii) $\iff$ (iii) and (ii) $\iff$ (iv) are clear. $\square$

Clearly linear combinations of operators with controlled support again have controlled support, as does a product of operators with controlled support. So the operators with controlled support form a unital $*$-algebra.

Definition 4.5.3. Denote by $\mathbb{C}_{\mathrm{cs}}[X]$ the subalgebra $\{T \in B(L^2X) \mid T \text{ has controlled support}\}$ of $B(L^2X)$. We will denote it $\mathbb{C}_{\mathrm{cs}}[X,\mu]$ when the measure is not clear from the context.

Remark 4.5.4. In the case that X is uniformly locally finite, the algebra $\mathbb{C}_{\mathrm{cs}}[X]$ is the same as in Definition 2.4.8.

Definition 4.5.5. Let $T \in \mathbb{C}_{\mathrm{cs}}[X]$, let E be a controlled set and let $\varepsilon > 0$. We say that T is an *operator in* (E,ε)*-blocks* if it is supported on $\bigsqcup_i A_i \times A_i$ where (A_i) is some (E,ε)-blocking collection. We say that T is an *operator in blocks* if it is supported on some controlled set in blocks.

For a measurable subset $U \subseteq X$, denote by $\pi_U\colon L^2X \to L^2U$ the restriction map and by $i_U\colon L^2U \to L^2X$ extension by 0. An operator T that is supported on $\bigsqcup_i A_i \times A_i$ can be written as $T = \sum_i T_i$, where $T_i = \pi_{A_i} T i_{A_i}\colon L^2A_i \to L^2A_i$, using Lemma 4.5.2. This decomposition makes it very convenient to do computations with operators in blocks. Moreover, every operator in $\mathbb{C}_{\mathrm{cs}}[X]$ can be written as a finite sum of operators in blocks:

Lemma 4.5.6. *Let $T \in \mathbb{C}_{\mathrm{cs}}[X]$. We can write T as a finite sum $T_1 + \ldots + T_N$ where each $T_i \in \mathbb{C}_{\mathrm{cs}}[X]$ is an operator in blocks.*

Proof. Let T be supported on the measurable and symmetric controlled set E. Let (A_i) be a blocking collection as in the proof of Lemma 4.4.4 and consider the same colouring on I as in that proof. Let

$$T_k = \sum_{i \text{ with colour } k} \pi_{A_i} T.$$

Then T_k is controlled on $\sqcup_{i \text{ with colour } k} E_{A_i} \times E_{A_i}$, so it is an operator in blocks. Moreover we have $T = T_1 + \ldots + T_N$. $\square$

We define some spaces of functions on X:

Definition 4.5.7. (i) Let L^2_bX denote the L^2-functions on X with bounded support.

(ii) Let LL^2X denote the measurable functions on X that are locally L^2: for each $\varphi \in LL^2X$ and each bounded measurable U, we have $\int_U |\varphi|^2 < \infty$.

Lemma 4.5.8. *The dual of L^2_bX can be identified with LL^2X.*

Proof. For any $\xi \in L^2_b X$ and $\eta \in LL^2 X$, the inner product $\langle \eta, \xi \rangle$ is finite, so η defines a function in $(L^2_b X)^*$.

Conversely, let $\varphi \in (L^2_b X)^*$. For any bounded measurable U, the function φ can be pulled back along the inclusion $L^2 U \hookrightarrow L^2_b X$ to an element of $(L^2 U)^*$, which may be viewed as a function $\eta_U \in L^2 U$. If $U \subseteq V$ are bounded and measurable, then we find that $\eta_U = \eta_V|_U$. Therefore, the η_U may be glued together to a function $\eta \in LL^2 X$. $\square$

If $T \in B(L^2 X)$ has controlled support, it sends $L^2_b X$ to $L^2_b X$, and so does the dual T^*. Taking the dual of $T^* \colon L^2_b(X) \to L^2_b(X)$ gives a map $LL^2 X \to LL^2 X$ which we denote by T again. Now define the linear map $\Phi \colon \mathbb{C}_{\mathrm{cs}}[X] \to LL^2 X$ by $\Phi(T) = T(\mathbb{1}_X) \in LL^2 X$. Explicitly, if T is supported on a controlled set E then we have

$$\Phi(T)(x) = (T\mathbb{1}_{E_x})(x).$$

For all $T, S \in \mathbb{C}_{\mathrm{cs}}[X]$ we have $\Phi(ST) = S\Phi(T)$.

Lemma 4.5.9. *Let $T \in \mathbb{C}_{\mathrm{cs}}[X]$ be such that $\Phi(T)$ is bounded. Then we can write $T = T_1 + \ldots + T_N$ where the T_k are operators in blocks and $\Phi(T_k)$ is bounded.*

Proof. Construct T_k as in the proof of Lemma 4.5.6. Then the $\Phi(T_k)$ are bounded. $\square$

4.6 Geometric property (T)

Definition 4.6.1. A representation of $\mathbb{C}_{\mathrm{cs}}[X]$ consists of a Hilbert space $\mathcal{H}$ and a unital $*$-homomorphism $\rho \colon \mathbb{C}_{\mathrm{cs}}[X] \to B(\mathcal{H})$. If no confusion arises we will omit the map ρ from the notation.

The standard representation of $\mathbb{C}_{\mathrm{cs}}[X]$ is the inclusion $\mathbb{C}_{\mathrm{cs}}[X] \hookrightarrow B(L^2 X)$. Note that we do not ask the map ρ to be continuous with respect to the norm on $\mathbb{C}_{\mathrm{cs}}[X]$, inherited from $B(L^2 X)$.

Definition 4.6.2. Let $T, S \in \mathbb{C}_{\mathrm{cs}}[X]$.

(i) We write $T \geq S$ if this inequality holds in $B(L^2 X)$.

(ii) We write $T \geq_{\max} S$ if for every representation $(\rho, \mathcal{H})$ of $\mathbb{C}_{\mathrm{cs}}[X]$, we have the inequality $\rho(T) \geq \rho(S)$.

(iii) Let $\|T\|_{L^2}$ be the norm of T in $B(L^2 X)$.

(iv) Define $\|T\|_{\max} = \sup\|\rho(T)\|$, where the supremum is over all representations $(\rho, \mathcal{H})$. We will see in Corollary 4.6.4 that this is always finite, and hence it also defines a norm on $\mathbb{C}_{\mathrm{cs}}[X]$.

Lemma 4.6.3. *Let $T \in \mathbb{C}_{\mathrm{cs}}[X]$ be an operator in blocks.*

(i) If $T \geq 0$, then $T \geq_{\max} 0$.

(ii) We have $\|T\|_{L^2} = \|T\|_{\max}$.

Proof. (i) Suppose T is supported on $\bigsqcup_i A_i \times A_i$ where (A_i) is a blocking collection. Write $T = \sum_i T_i$ with $T_i \colon L^2 A_i \to L^2 A_i$. Then $T^{\frac{1}{2}} = \sum_i T_i^{\frac{1}{2}}$ is again an operator in blocks. Now for any representation $(\rho, \mathcal{H})$ we have $\rho(T) = \rho(T^{\frac{1}{2}})^2 \geq 0$, so we have $T \geq_{\max} 0$.

(ii) Clearly we have $\|T\|_{L^2} \leq \|T\|_{\max}$. Note that T^*T is again an operator in blocks. We have $0 \leq T^*T \leq \|T\|_{L^2}^2$, and by part (i), it follows that $0 \leq_{\max} T^*T \leq_{\max} \|T\|_{L^2}^2$. So for any representation $(\rho, \mathcal{H})$, we have $0 \leq \rho(T^*T) \leq \|T\|_{L^2}^2$. So $\|\rho(T)\| = \|\rho(T)^*\rho(T)\|^{\frac{1}{2}} = \|\rho(T^*T)\|^{\frac{1}{2}} \leq \|T\|_{L^2}$, and hence $\|T\|_{L^2} = \|T\|_{\max}$.

□

If T is an operator in blocks we will simply write $\|T\|$ for both the norms $\|T\|_{L^2}$ and $\|T\|_{\max}$.

Corollary 4.6.4. *For any $T \in \mathbb{C}_{\mathrm{cs}}[X]$, the maximal norm $\|T\|_{\max}$ is finite.*

Proof. This follows directly from Lemma 4.6.3 and Lemma 4.5.6. □

Therefore $\|\cdot\|_{\max}$ defines a norm and we can define the maximal C^*-algebra:

Definition 4.6.5. The maximal C^*-algebra $C^*_{\max}(X)$ is defined as the completion of $\mathbb{C}_{\mathrm{cs}}[X]$ with respect to the maximal norm $\|\cdot\|_{\max}$.

For a uniformly locally finite space X the C^*-algebra $C^*_{\max}(X)$ is the same as the one defined in [58] (there it is called $C^*_{u,\max}(X)$). For $f \in L^\infty X$ denote by $M_f \in B(L^2X)$ the multiplication operator by f. Similar to [58], we define the subspace of the constant vectors.

Definition 4.6.6. Let $\mathcal{H}$ be a non-degenerate representation of $\mathbb{C}_{\mathrm{cs}}[X]$, and let $v \in \mathcal{H}$. We call v a *constant vector* if $Tv = M_{\Phi(T)}v$ for all operators $T \in \mathbb{C}_{\mathrm{cs}}[X]$ for which $\Phi(T)$ is bounded. Here Φ is as defined at the end of Section 4.5. The constant vectors form a closed subspace $\mathcal{H}_c$ of $\mathcal{H}$.

Since we can replace T by $T - M_{\Phi(T)}$, it is enough to ask that $Tv = 0$ for all T with $\Phi(T) = 0$. Now we are ready to define geometric property (T).

Definition 4.6.7. Let X be a coarse space with bounded geometry and let μ be a uniformly bounded measure for which a gordo set exists. We say that (X, μ) satisfies *geometric property (T)* if there are a controlled set E and constants $\varepsilon, \gamma > 0$ such that for every non-degenerate representation $(\rho, \mathcal{H})$ and every unit vector $v \in \mathcal{H}_c^\perp$ there is an operator T in (E, ε)-blocks with $\Phi(T)$ bounded and $\left\|(T - M_{\Phi(T)})v\right\| \geq \gamma\|T\|$.

In the definition, it is necessary to ask that $\Phi(T)$ is a bounded function, because otherwise $M_{\Phi(T)}$ would not be defined. By considering the operator $T - M_{\Phi(T)}$, we see that we can equivalently ask there to be an operator T in (E, ε)-blocks with $\Phi(T) = 0$ and $\|Tv\| \geq \gamma\|T\|$.

If X is uniformly locally finite with a generating controlled set, and μ is the counting measure, we retrieve the definition of [58] (cf. Proposition 3.1 in that paper):

Lemma 4.6.8. *Let X be a uniformly locally finite space for which there is a generating controlled set, and let μ be the counting measure. The following are equivalent:*

(i) The space (X, μ) satisfies geometric property (T).

(ii) There exists a controlled generating set E and a constant $\gamma > 0$ such that for any representation $\mathcal{H}$ and unit vector $v \in \mathcal{H}_c^\perp$, there is an operator $T \in \mathbb{C}_{\mathrm{cs}}[X]$ with support in E such that

$$\left\|(T - M_{\Phi(T)})v\right\| > \gamma \sup_{x,z} |T_{xz}|,$$

where T_{xz} denotes the value at (x, z) of the matrix corresponding to the operator T.

Proof. We certainly have $\|T\|_{L^2} \geq \sup_{x,z} |T_{xz}|$. With this, the direction (i) $\Longrightarrow$ (ii) is trivial, since we may enlarge the controlled set E to make it generating.

Now suppose (ii) is true, take E, γ satisfying the condition, and let $\varepsilon = 1$. Without loss of generality, we may assume E to be symmetric. There is an integer N such that $\#E_x \leq N$ for each $x \in X$. Let $\mathcal{H}$ be a representation and $v \in \mathcal{H}_c^\perp$ a unit vector. We know there is $T \in \mathbb{C}_{\mathrm{cs}}[X]$ supported on E such that $\left\|(T - \Phi(T))v\right\| > \gamma \sup_{x,z} |T_{xz}|$. For each pair $(x,z) \in E$ there are fewer than $4N$ other pairs $(x', z') \in E$ with $\{x,z\} \cap \{x', z'\} \neq \emptyset$. By a graph colouring argument we may write $T = T_1 + \ldots + T_{4N}$ as a sum of operators controlled on E with for each k: if $(x,z) \in E$ and $(x', z') \in E$ are different pairs where T_k is supported, then $\{x,z\} \cap \{x', z'\} = \emptyset$. Moreover we get $\sup_{x,z} |T_{xz}| = \sup_k \|T_k\|_{L^2}$. We then have

$$\left\|\sum_k (T_k - M_{\Phi(T_k)})v\right\| = \left\|(T - M_{\Phi(T)})v\right\| \geq \gamma \sup_{x,z} |T_{xz}| = \gamma \sup_k \|T_k\|_{L^2},$$

hence there is a k with $\left\|(T_k - M_{\Phi(T_k)})v\right\| \geq \frac{\gamma}{4N} \|T_k\|_{L^2}$, proving (i). □

4.7 Laplacians

In this section we will give a different characterisation of geometric property (T) using spectral properties of Laplacians.

Definition 4.7.1. Let E be a measurable symmetric controlled set. Define the *Laplacian* $\Delta^E \in B(L^2X)$ as

$$\Delta^E(\xi)(x) = \int_{E_x} (\xi(x) - \xi(y)) d\mu(y).$$

The Laplacian corresponding to E is obviously supported on E, in particular we have $\Delta^E \in \mathbb{C}_{\mathrm{cs}}[X]$. The Laplacians are positive operators, as is shown by the following lemma.

Lemma 4.7.2. *For any measurable symmetric controlled set E we have $0 \leq_{\max} \Delta^E \leq_{\max} 2M$, where $M = \sup_x \mu(E_x)$.*

Proof. A straightforward computation shows that for all measurable symmetric controlled sets F, we have

$$\begin{aligned}
\langle \Delta^F \xi, \xi \rangle &= \int_X \overline{\Delta^F \xi(x)} \xi(x) d\mu(x) \\
&= \int_X \int_{F_x} \overline{(\xi(x) - \xi(y))} \xi(x) d\mu(y) d\mu(x) \\
&= \frac{1}{2} \int_X \int_{F_x} \left(\overline{(\xi(x) - \xi(y))} \xi(x) + \overline{(\xi(y) - \xi(x))} \xi(y) \right) d\mu(y) d\mu(x) \\
&= \frac{1}{2} \int_F |\xi(x) - \xi(y)|^2 d(\mu \times \mu)(x,y) \\
&\geq 0,
\end{aligned}$$

so $\Delta^F \geq 0$. Since $|\xi(x)-\xi(y)|^2 \leq 2|\xi(x)|^2+2|\xi(y)|^2$ we also have

$$\langle \Delta^F \xi, \xi\rangle \leq \int_F (|\xi(x)|^2+|\xi(y)|^2) d(\mu\times\mu)(x,y) = 2\int_F |\xi(x)|^2 d(\mu\times\mu)(x,y)$$

by symmetry. This is equal to $2\int_F |\xi(x)|^2 \mu(F_x) d\mu(x)$, which is at most $2\|\xi\|^2 \sup_x \mu(F_x)$, so we have $\left\|\Delta^F\right\|_{L^2} \leq 2\sup_x \mu(F_x)$. Moreover, if F_1, F_2 are disjoint measurable symmetric controlled sets, we see directly that $\Delta^{F_1\cup F_2} = \Delta^{F_1}+\Delta^{F_2}$.

Let E be any measurable symmetric controlled set. By Lemma 4.4.4, there are symmetric controlled sets in blocks $E_1,\dots,E_N$ whose union contains E. Let $E_1',\dots,E_N'$ be pairwise disjoint measurable sets such that $E_k' \subseteq E_k$ for each k and $E = \bigsqcup_k E_k'$. For each k, the operator $\Delta^{E_k'}$ is an operator in blocks, so by Lemma 4.6.3 we have $0 \leq_{\max} \Delta^{E_k'} \leq_{\max} 2\sup_x \mu((E_k')_x)$. Taking the sum from $k=1$ to N, we get $0 \leq_{\max} \Delta^E \leq_{\max} 2\sup_x \mu(E_x)$. □

Lemma 4.7.3. *Let E be a controlled set in blocks and let $T \in \mathbb{C}_{\mathrm{cs}}[X]$ be a positive operator supported on E such that $\Phi(T)=0$. Then*

$$T \leq_{\max} \frac{\|T\|}{\inf_x(\mu(E_x))}\Delta^E,$$

where the infimum is taken over all x with $E_x \neq \emptyset$.

Proof. Write $E = \bigsqcup_i A_i \times A_i$ and $T = \sum_i T_i$ with $T_i\colon L^2A_i \to L^2A_i$. It is enough to prove the inequality $T_i \leq \frac{\|T_i\|}{\mu(A_i)}\Delta^{E_i}$ for each i, where $E_i = A_i\times A_i$. The space L^2A_i can be written as the direct sum of the constant functions and the functions with zero integral. For any $\xi\in L^2A_i$ write $\xi = \xi_c+\xi_d$ where ξ_c is constant and $\int \xi_d = 0$. Then $T_i\xi = T_i\xi_d$ and $\Delta^{E_i}\xi = \mu(A_i)\xi_d$. So we have $\langle T_i\xi,\xi\rangle \leq \|T_i\|\cdot\|\xi_d\|^2 = \langle \frac{\|T_i\|}{\mu(A_i)}\Delta^{E_i}\xi,\xi\rangle$ showing that $T_i \leq \frac{\|T_i\|}{\mu(A_i)}\Delta^{E_i}$. □

Since $\Phi(\Delta^E)=0$, we have $\Delta^E v = 0$ for all $v\in\mathcal{H}_c$, where $\mathcal{H}$ is a representation of $\mathbb{C}_{\mathrm{cs}}[X]$. Moreover, we have the following lemma, which shows the converse.

Lemma 4.7.4. *Let $(\rho,\mathcal{H})$ be a representation of $\mathbb{C}_{\mathrm{cs}}[X]$. Then*

$$\mathcal{H}_c = \cap_E \ker(\rho(\Delta^E))$$

where the intersection ranges over all the measurable symmetric controlled sets E.

Proof. Let $v\in\cap_E\ker(\rho(\Delta^E))$ and let $T\in\mathbb{C}_{\mathrm{cs}}[X]$ such that $\Phi(T)$ is bounded, we will show that $Tv = M_{\Phi(T)}v$, showing that $v\in\mathcal{H}_c$. By Lemma 4.5.9, we can assume that T is an operator in blocks. We can also assume that $\Phi(T)=0$ by considering the operator $T-\Phi(T)$. By Lemma 4.7.3, there is a measurable symmetric controlled set E and a constant c such that $T^*T \leq_{\max} c\Delta^E$, since T^*T is a positive operator in blocks. Then $\Delta^E v = 0$ and we get $\|Tv\|^2 = \langle T^*Tv, v\rangle \leq c\langle\Delta^E v, v\rangle = 0$, so $Tv=0$. □

Definition 4.7.5. Let $T\in C^*_{\max}(X)$ be a positive operator. We say that T *has spectral gap* if there is a constant $\gamma>0$ such that $\sigma_{\max}(T)\subseteq\{0\}\cup[\gamma,\infty)$, where $\sigma_{\max}$ denotes the spectrum of T in the maximal C^*-algebra.

Now we can prove the generalisation of [58, Proposition 5.2].

Proposition 4.7.6. *Let X be a coarse space of bounded geometry and let μ be a uniformly bounded measure for which a gordo set exists. Then (X, μ) has geometric property (T) if and only if there exists a measurable symmetric controlled set E such that for each representation $(\rho, \mathcal{H})$ we have $\mathcal{H}_c = \ker(\rho(\Delta^E))$ and Δ^E has spectral gap.*

Proof. Suppose X has property (T). Let E be a controlled set and $\varepsilon, \gamma > 0$ constants such that for every representation $\mathcal{H}$ and every unit vector $v \in \mathcal{H}_c^\perp$, there is an operator T in (E, ε)-blocks such that $\Phi(T) = 0$ and $\|Tv\| \geq \gamma \|T\|$. We will show that for each representation $(\rho, \mathcal{H})$, we have $\mathcal{H}_c = \ker(\rho(\Delta^E))$ and Δ^E has spectral gap.

Let $v \in \mathcal{H}_c^\perp$ be a unit vector. Let T be an operator in (E, ε)-blocks with $\Phi(T) = 0$ and $\|Tv\| \geq \gamma \|T\|$. Let T be supported on the blocks (A_i) and let $E' = \bigsqcup_i A_i \times A_i \subseteq E$. By Lemma 4.7.3, we have $T^*T \leq \frac{\|T\|^2}{\inf_i \mu(A_i)} \Delta^{E'} \leq \frac{\|T\|^2}{\varepsilon} \Delta^E$. Now

$$\gamma^2 \|T\|^2 \leq \|Tv\|^2 = \langle T^*Tv, v\rangle \leq \frac{\|T\|^2}{\varepsilon} \langle \Delta^E v, v\rangle \leq \frac{\|T\|^2}{\varepsilon} \left\| \Delta^E v \right\|,$$

so $\left\|\Delta^E v\right\| \geq \gamma^2 \varepsilon$. This shows that $\ker(\rho(\Delta^E)) \cap \mathcal{H}_c^\perp = \{0\}$, so $\ker(\rho(\Delta^E)) = \mathcal{H}_c$. Moreover, it shows that $\sigma(\rho(\Delta^E)) \subseteq \{0\} \cup [\gamma^2\varepsilon, \infty)$. Since this holds for every representation $(\rho, \mathcal{H})$, we get $\sigma_{\max}(\Delta^E) \subseteq \{0\} \cup [\gamma^2 \varepsilon, \infty)$, hence Δ^E has spectral gap.

Now suppose that E is a measurable symmetric controlled set such that for each representation $(\rho, \mathcal{H})$ we have $\mathcal{H}_c = \ker(\rho(\Delta^E))$ and Δ^E has spectral gap, say $\sigma_{\max}(\Delta^E) \subseteq \{0\} \cup [\gamma, \infty)$. By Lemma 4.4.4 there are controlled sets in blocks $E_1, \ldots, E_N$ whose union contains E. Let $\varepsilon = \min_k \inf_x \mu((E_k)_x) > 0$, where k ranges from 1 to N and x ranges over all elements in X with $(E_k)_x \neq \emptyset$. Now let $(\rho, \mathcal{H})$ be a representation and let $v \in \mathcal{H}_c^\perp$ be a unit vector. Then $v \in \ker(\rho(\Delta^E))^\perp$ so $\langle \Delta^E v, v\rangle \geq \gamma$. We also have $\Delta^E \leq \Delta^{E_1} + \ldots + \Delta^{E_N}$ so there is $1 \leq k \leq N$ such that $\langle \Delta^{E_k} v, v\rangle \geq \frac{\gamma}{N}$. Now Δ^{E_k} is an (E, ε)-operator in blocks and $\left\|\Delta^{E_k} v\right\| \geq \frac{\gamma}{N}$, so we see that X has geometric property (T). $\square$

In [58, Proposition 5.2] it is actually shown that $\mathcal{H}_c = \ker(\rho(\Delta^E))$ holds for all generating symmetric controlled sets E and that if X has geometric property (T), then Δ^E has spectral gap for all generating symmetric controlled sets E. In our case, we have a similar result, for which we first need another lemma and its corollary.

Lemma 4.7.7. *Let E be a gordo set and F a symmetric measurable controlled set containing E and satisfying $F \circ E = E \circ F$. Let $f \in L^\infty(X)$ be the function $f(x) = \mu((E \circ F)_x)$. Then there is a constant δ such that*

$$\delta \Delta^{F \circ F} \leq_{\max} (2M_f - \Delta^{E\circ F}) M_{\frac{1}{f}} \Delta^{E \circ F}.$$

Proof. For every $\xi \in L^2 X$ and $x \in X$, we have

$$(2M_f - \Delta^{E \circ F})\xi(x) = f(x)\xi(x) + \int_{(E\circ F)_x} \xi(y) d\mu(y).$$

Then we have

$$\begin{aligned}
&(2M_f - \Delta^{E\circ F}) M_{\frac{1}{f}} \Delta^{E\circ F} \xi(x) \\
=&\Delta^{E\circ F}\xi(x) + \int_{(E\circ F)_x} \frac{1}{f(y)} \Delta^{E\circ F}\xi(y) d\mu(y)
\end{aligned}$$

$$
\begin{aligned}
=&f(x)\xi(x) - \int_{(E\circ F)_x} \xi(y)d\mu(y) + \int_{(E\circ F)_x} \xi(y)d\mu(y) - \int_{(E\circ F)_x} \frac{1}{f(y)} \int_{(E\circ F)_y} \xi(z)d\mu(z)d\mu(y) \\
=&f(x)\xi(x) - \int_{(E\circ F)_x^{\circ 2}} \alpha(x,z)\xi(z)d\mu(z),
\end{aligned}
$$

where

$$
\alpha(x,z) = \int_{(E\circ F)_x \cap (E\circ F)_z} \frac{1}{f(y)} d\mu(y).
$$

Let $z \in (F\circ F)_x$. There is $y \in X$ with $(x,y) \in F$ and $(y,z) \in F$. It follows that $E_y \subseteq (E\circ F)_x \cap (E\circ F)_z$, so $\alpha(x,z) \geq \int_{E_y} \frac{1}{f(y')} d\mu(y')$. Since E is gordo and f is bounded from above, there is a constant δ such that $\alpha(x,z) \geq \delta$ for all $z \in (F\circ F)_x$. Now define

$$
\beta(x,z) = \begin{cases} \alpha(x,z) - \delta & \text{if } (x,z) \in F\circ F; \\ \alpha(x,z) & \text{otherwise.} \end{cases}
$$

Then β is symmetric, $\beta(x,z) \geq 0$ for all $(x,z) \in X \times X$ and

$$
(2M_f - \Delta^{E\circ F})M_{\frac{1}{f}}\Delta^{E\circ F} = \delta\Delta^{F\circ F} + T,
$$

where $T \in \mathbb{C}_{cs}[X]$ is the operator defined by

$$
T\xi(x) = \int_{(E\circ F)_x^{\circ 2}} \beta(x,y)(\xi(x) - \xi(y))d\mu(y).
$$

It remains to show that $T \geq_{\max} 0$. This is proved similarly to Lemma 4.7.2. Let $E_1, \ldots, E_N$ be controlled sets in blocks whose union contains $(E\circ F)^{\circ 2}$. Let $E_1', \ldots, E_N'$ be measurable symmetric subsets of $E_1, \ldots, E_N$ respectively, that are disjoint from each other and whose union is $(E\circ F)^{\circ 2}$. For $1 \leq k \leq N$ define $T_k \in \mathbb{C}_{\mathrm{cs}}[X]$ by

$$
T_k\xi(x) = \int_{(E_k')_x} \beta(x,y)(\xi(x) - \xi(y))d\mu(y).
$$

An easy computation shows that

$$
\langle T_k\xi, \xi\rangle = \frac{1}{2}\int_{E_k'} \beta(x,y)(\xi(x) - \xi(y))^2 d(\mu\times\mu)(x,y),
$$

so $T_k \geq 0$. Since T_k is an operator in blocks, we get $T_k \geq_{\max} 0$ by Lemma 4.6.3. So we have $T = \sum_{k=1}^N T_k \geq_{\max} 0$. ☐

Corollary 4.7.8. *Let E be a gordo set and F a symmetric measurable controlled set satisfying $F\circ E = E\circ F$ and $E \subseteq F$. Then for every representation $(\rho, \mathcal{H})$, we have $\ker(\rho(\Delta^{F\circ F})) = \ker(\rho(\Delta^{E\circ F}))$. Moreover, $\Delta^{F\circ F}$ has spectral gap if and only if $\Delta^{E\circ F}$ has spectral gap.*

Proof. Since $E \subseteq F$ we have $E\circ F \subseteq F\circ F$, and therefore $\Delta^{E\circ F} \leq_{\max} \Delta^{F\circ F}$. It follows that $\ker(\rho(\Delta^{F\circ F})) \subseteq \ker(\rho(\Delta^{E\circ F}))$ and that if $\Delta^{E\circ F}$ has spectral gap, then $\Delta^{F\circ F}$ also has spectral gap. The other direction follows from Lemma 4.7.7. ☐

Proposition 4.7.9. *Let X be a coarse space of bounded geometry and let μ be a uniformly bounded measure. Let E be a gordo set that generates the coarse structure on X. Let $E' = E^{\circ 3}$. Then for every representation $(\rho, \mathcal{H})$ of $\mathbb{C}_{\mathrm{cs}}[X]$, we have $\ker(\rho(\Delta^{E'})) = \mathcal{H}_c$, and X has geometric property (T) if and only if $\Delta^{E'}$ has spectral gap.*

Proof. We apply Corollary 4.7.8 with $F = E^{\circ n}$ for $n \geq 2$. By induction it follows that for all $n \geq 3$ we have $\ker(\rho(\Delta^{E^{\circ n}})) = \ker(\rho(\Delta^{E'}))$ and that $\Delta^{E^{\circ n}}$ has spectral gap if and only if $\Delta^{E'}$ has spectral gap.

If X has geometric property (T), then there is a controlled set G such that $\ker(\rho(\Delta^G)) = \mathcal{H}_c$ for every representation $(\rho, \mathcal{H})$ and Δ^G has spectral gap. Since E is generating, there is some n such that $G \subseteq E^{\circ n}$. Then $\Delta^G \leq_{\max} \Delta^{E^{\circ n}}$. So $\ker(\rho(\Delta^{E'})) = \ker(\rho(\Delta^{E^{\circ n}})) = \mathcal{H}_c$ for every representation $(\rho, \mathcal{H})$ and $\Delta^{E^{\circ n}}$ and $\Delta^{E'}$ have spectral gap. □

We will now give a third characterisation of geometric property (T) based more directly on the C*-algebra $C^*_{\max}(X)$. Note that $\ker(\Phi)$ is a left ideal in $\mathbb{C}_{\mathrm{cs}}[X]$. Let $I_c(X) = C^*_{\max}(X)\ker(\Phi)$ be the left ideal in $C^*_{\max}(X)$ generated by $\ker(\Phi)$.

Proposition 4.7.10. *Let X be a coarse space of bounded geometry and let μ be a uniformly bounded measure for which a gordo set exists. Then (X, μ) has geometric property (T) if and only if there is a projection $p \in C^*_{\max}(X)$ generating $I_c(X)$.*

Proof. First suppose that (X, μ) has geometric property (T). Let $(\rho, \mathcal{H})$ be a faithful unital representation of $C^*_{\max}(X)$. By Proposition 4.7.6 there is a controlled set E such that $\ker(\rho(\Delta^E)) = \mathcal{H}_c$ and Δ^E has spectral gap. By functional calculus there is a projection $p \in I_c(X)$ such that $pv = 0$ for all $v \in \mathcal{H}_c$ and $pv = v$ for all $v \in \mathcal{H}_c^{\perp}$. Now for any $t \in I_c(X)$ we have $tv = 0$ for all $v \in \mathcal{H}_c$, hence $t = tp$. So p generates $I_c(X)$.

Conversely, suppose $p \in C^*_{\max}(X)$ is a projection that generates $I_c(X)$. Then we can write $p = \sum_{j=1}^{K} t_j S_j$ where K is an integer and $t_j \in C^*_{\max}(X)$ and $S_j \in \ker(\Phi) \subseteq \mathbb{C}_{\mathrm{cs}}[X]$. Choose $T'_j \in \mathbb{C}_{\mathrm{cs}}[X]$ with $\left\| t_j - T'_j \right\| \leq \frac{1}{2K\|S_j\|}$. Let $T = \sum_{j=1}^{K} T_j S_j$, then $T \in \mathbb{C}_{\mathrm{cs}}[X]$, and $\Phi(T) = 0$, and $\|T - p\| \leq \sum_{j=1}^{K} \left\| t_j - T'_j \right\| \cdot \|S\|_j \leq \frac{1}{2}$. Write $T = T_1 + \ldots + T_N$ as a sum of operators in blocks and choose a controlled set E and a constant $\varepsilon > 0$ such that they are all in (E, ε)-blocks. Now let $(\rho, \mathcal{H})$ be any representation. For any $v \in \ker(p)$ we have $sv = 0$ for all $s \in I_c(X)$, hence $v \in \mathcal{H}_c$. So $\ker(p) = \mathcal{H}_c$. Now let $v \in \mathcal{H}_c^{\perp}$ be a unit vector. Then $pv = v$ so $\|Tv\| \geq \frac{1}{2}$. Then there is $1 \leq k \leq N$ such that $\|T_k v\| \geq \frac{1}{2N}$, showing that (X, μ) has geometric property (T). □

4.8 Coarse invariance

In this section, we will prove that geometric property (T), as defined in Definition 4.6.7, does not depend on the measure μ chosen, and moreover, that it is a coarse invariant. Our method is inspired by the one employed in [58]. Let X be a coarse space with bounded geometry, and let μ be a uniformly bounded measure for which a gordo set exists. We will start by constructing a uniformly locally finite space Y that is coarsely dense in X, with an appropriate measure ν. We will show a correspondence between representations of $\mathbb{C}_{\mathrm{cs}}[Y]$ and some of the representations of $\mathbb{C}_{\mathrm{cs}}[X]$. This will allow us to see $C^*_{\max}(Y)$ as a subalgebra of $C^*_{\max}(X)$. Then we can apply the criterion for geometric property (T) given in Proposition 4.7.10 to conclude that (X, μ) has geometric property (T) if and only if (Y, ν) has geometric property (T). After this we will show that geometric property

(T) does not depend on the measure as long as the space is uniformly locally finite. Finally, we will conclude that if X is coarse equivalent to another space X', with a uniformly bounded measure μ' for which a gordo set exists, then (X, μ) has geometric property (T) if and only if (X', μ') has geometric property (T).

By Lemma 4.4.2, there is a blocking collection with union X. Write $X = \bigsqcup_{y \in Y} U_y$, where (U_y) is a blocking collection and y is a point in U_y (the notation U_y is not to be confused with the notation E_x introduced in Definition 4.2.1(ii)). Note that Y is coarsely dense in X and that $\mu(U_y)$ is bounded from above and below, uniformly in y. Let ν be the measure on Y defined by $\nu(\{y\}) = \mu(U_y)$. We will first show that (X, μ) has property (T) if and only if (Y, ν) has property (T).

Lemma 4.8.1. *Let X be a coarse space with bounded geometry and let μ be a uniformly bounded measure on X for which a gordo set exists. Let Y be a coarsely dense, uniformly locally finite subspace of X with a measure ν. Suppose $X = \bigsqcup_y U_y$ can be written as a disjoint union of measurable sets with $y \in U_y$ and $\nu(y) = \mu(U_y)$ and such that $\bigsqcup_y U_y \times U_y$ is a controlled set. Then (X, μ) has property (T) if and only if (Y, ν) has property (T).*

We need some preparation before we can prove Lemma 4.8.1. For $x \in X$, let $y(x) \in Y$ be the unique point such that $x \in U_y$. Define the linear maps $a\colon L^2X \to L^2(Y, \nu)$ and $b\colon L^2(Y, \nu) \to L^2X$ by $a\xi(y) = \frac{1}{\nu(\{y\})} \int_{U_y} \xi$ and $b\eta(x) = \eta(y(x))$. Note that these are adjoint to each other and that ab is the identity on $L^2(Y, \nu)$. Let $A = ba$. This is a projection in $\mathbb{C}_{\mathrm{cs}}[X]$. The maps $\alpha\colon A\mathbb{C}_{\mathrm{cs}}[X]A \to \mathbb{C}_{\mathrm{cs}}[Y]$, $\alpha(T) = aTb$ and $\beta\colon \mathbb{C}_{\mathrm{cs}}[Y] \to A\mathbb{C}_{\mathrm{cs}}[X]A$, $\beta(S) = bSa$ are $*$-isomorphisms and inverse to each other. Now if $\mathcal{H}^X$ is a representation of $\mathbb{C}_{\mathrm{cs}}[X]$, then $\mathcal{H}^Y = A\mathcal{H}^X$ is a representation of $A\mathbb{C}_{\mathrm{cs}}[x]A$ and via β also a representation of $\mathbb{C}_{\mathrm{cs}}[Y]$. Conversely, every representation of $\mathbb{C}_{\mathrm{cs}}[Y]$ gives rise to a representation of $\mathbb{C}_{\mathrm{cs}}[X]$.

Lemma 4.8.2. *Let $(\rho^Y, \mathcal{H}^Y)$ be a representation of $\mathbb{C}_{\mathrm{cs}}[Y]$. Then there is a representation $(\rho^X, \mathcal{H}^X)$ such that $A\mathcal{H}^X = \mathcal{H}^Y$, and such that ρ^Y equals the restriction of $\rho^X \circ \beta$ to $A\mathcal{H}^X$. Here $\mathcal{H}^X$ is a completion of a quotient of the algebraic tensor product $\mathbb{C}_{\mathrm{cs}}[X] \odot \mathcal{H}^Y$ that we will denote by $\mathbb{C}_{\mathrm{cs}}[X] \otimes \mathcal{H}^Y$.*

Proof. First consider the algebraic tensor product $\mathbb{C}_{\mathrm{cs}}[X] \odot \mathcal{H}^Y$. We equip this with the sesquilinear form defined on simple tensors by

$$\langle T \odot v, T' \odot v' \rangle = \langle \alpha(A(T')^* TA)v, v' \rangle$$

and extended to $(\mathbb{C}_{\mathrm{cs}}[X] \odot \mathcal{H}^Y)^2$ by sesquilinearity. It is easy to see that this is well defined and defines a conjugate symmetric sesquilinear form. We will now show that it is positive semi-definite.

For any operator $T \in \mathbb{C}_{\mathrm{cs}}[X]$ there is an integer M, such that for every $y \in Y$ there are at most M different $y' \in Y$ such that $\pi_{U_y} T i_{U_{y'}} \neq 0$ or $\pi_{U_{y'}} T i_{U_y} \neq 0$. By a graph colouring argument we may write $T = T_1 + \ldots + T_N$, such that for every $1 \leq k \leq N$, and every $y \in Y$, there is at most one y' such that $\pi_{U_y} T_k i_{U_{y'}} \neq 0$ or $\pi_{U_{y'}} T_k i_{U_y} \neq 0$.

It follows that each element of $\mathbb{C}_{\mathrm{cs}}[X] \odot \mathcal{H}^Y$ is of the form $\sum_{k=1}^N T_k \odot v_k$ with $T_k \in \mathbb{C}_{\mathrm{cs}}[X], v_k \in \mathcal{H}^Y$ and the T_k satisfy that for every y there is at most one y' such that $\pi_{U_y} T i_{U_{y'}} \neq 0$ or $\pi_{U_{y'}} T i_{U_y} \neq 0$. Let this y' be $f_k(y)$ if it exists, otherwise put $f_k(y) = y$. Then each f_k is an involutive function (i.e. $f_k \circ f_k = \mathrm{id}_Y$) and T_k sends L^2U_y to $L^2U_{f_k(y)}$. Define

$$Q = \begin{pmatrix} 0 & T_1 & \dots & T_N \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in B(L^2X) \otimes M_{N+1}(\mathbb{C}) = B(L^2(X \times \{0, 1, \dots, N\})).$$

For $y \in Y$ define $V_y = U_y \times \{0\} \sqcup \left(\bigsqcup_{k=1}^N U_{f_k(y)} \times \{k\} \right) \subseteq X \times \{0, 1, \dots, N\}$. Then $X \times \{0, 1, \dots, N\} = \bigsqcup_y V_y$ and Q sends each L^2V_y to L^2V_y. Let

$$A \otimes I = \begin{pmatrix} A & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A \end{pmatrix} \in B(L^2(X \times \{0, \dots, N\})).$$

Then also $(A \otimes I)Q^*Q(A \otimes I)$ sends each L^2V_y to L^2V_y. It is also a positive element so we can define the square root $R = \left((A \otimes I)Q^*Q(A \otimes I)\right)^{\frac{1}{2}} \in (A \otimes I)B(L^2(X \times \{0, 1, \dots, N\}))(A \otimes I)$. It sends each L^2V_y to L^2V_y. Denote the entries of R by $R_{kl} \in AB(L^2X)A$. Then R_{kl} sends L^2U_y to $L^2U_{f_k(f_l(y))}$. So R_{kl} has controlled support, and we have $R_{kl} \in A\mathbb{C}_{\mathrm{cs}}[X]A$. We have $R^2 = (A \otimes I)Q^*Q(A \otimes I)$ and looking at the entry at (k, l) gives $\sum_{s=1}^N R_{ks}R_{sl} = AT_k^*T_lA$ and $R_{kl}^* = R_{lk}$ for all $1 \leq k, l \leq N$.

Now we can compute

$$\begin{aligned} &\left\langle \sum_k T_k \odot v_k, \sum_k T_k \odot v_k \right\rangle \\ =& \sum_{k,l} \langle \alpha(AT_k^*T_lA)v_l, v_k \rangle \\ =& \sum_{k,l,s} \langle \alpha(R_{ks}R_{sl})v_l, v_k \rangle \\ =& \sum_{k,l,s} \langle \alpha(R_{sl})v_l, \alpha(R_{sk})v_k \rangle \\ =& \sum_s \left\langle \sum_k \alpha(R_{sk})v_k, \sum_k \alpha(R_{sk})v_k \right\rangle \geq 0. \end{aligned}$$

So the sesquilinear form is semi-positive definite. Define $\mathcal{H}^X$ to be the Hilbert space obtained by dividing $\mathbb{C}_{\mathrm{cs}}[X] \odot \mathcal{H}^Y$ by the kernel of the semi-norm $\|v\| = \langle v, v \rangle^{\frac{1}{2}}$ and completing with respect to the resulting norm. Now we will define the representation $\rho^X : \mathbb{C}_{\mathrm{cs}}[X] \to B(\mathcal{H}^X)$.

Let $T \in \mathbb{C}_{\mathrm{cs}}[X]$. This acts on $\mathbb{C}_{\mathrm{cs}}[X] \odot \mathcal{H}^Y$ by $T \cdot (S \odot v) = TS \odot v$ and extending by linearity. To show that this defines a bounded map on $\mathcal{H}^X$ we need to show that there is a constant C depending on T such that $\|Tw\| \leq C \cdot \|w\|$ for all $w \in \mathbb{C}_{\mathrm{cs}}[X] \odot \mathcal{H}^Y$. For this we can assume that T is an operator in blocks, by Lemma 4.5.6. Then $S = (\|T\|^2 - T^*T)^{\frac{1}{2}}$ is an element in $\mathbb{C}_{\mathrm{cs}}[X]$ and we have for all $w \in \mathbb{C}_{\mathrm{cs}}[X] \odot \mathcal{H}^Y$:

$$\|T\|^2 \cdot \|w\|^2 - \|Tw\|^2 = \langle (\|T\|^2 - T^*T)w, w \rangle = \langle Sw, Sw \rangle \geq 0.$$

So $\|Tw\| \leq \|T\| \cdot \|w\|$, showing that the action of T on $\mathbb{C}_{\text{cs}}[X] \odot \mathcal{H}^Y$ defines a bounded linear map $\rho^X(T) \in B(\mathcal{H}^X)$. It is now easy to check that this defines a representation of $\mathbb{C}_{\text{cs}}[X]$ on $\mathcal{H}^X$.

Finally, note that we have an injection $\mathcal{H}^Y \hookrightarrow \mathbb{C}_{\text{cs}}[X] \odot \mathcal{H}^Y$ sending v to $1 \odot v$. This respects the sesquilinear form, so it induces an injection $\mathcal{H}^Y \hookrightarrow \mathcal{H}^X$. For a simple tensor $T \odot v \in \mathbb{C}_{\text{cs}}[X] \odot \mathcal{H}^Y$ it is easy to check that $A(T \odot v) = AT \odot v = 1 \odot \alpha(ATA)v$, hence $A(\mathbb{C}_{\text{cs}}[X] \odot \mathcal{H}^Y) = \mathcal{H}^Y$ and $A\mathcal{H}^X = \mathcal{H}^Y$. Then it is also easy to check that the restriction of $\rho^X \circ \beta$ to $A\mathcal{H}^X$ equals ρ^Y. □

Lemma 4.8.3. *The map $\beta\colon \mathbb{C}_{\text{cs}}[Y] \xrightarrow{\sim} A\mathbb{C}_{\text{cs}}[X]A$ extends to an isometry*

$$\beta\colon C^*_{\max}(Y) \xrightarrow{\sim} AC^*_{\max}(X)A.$$

Proof. Let $S \in \mathbb{C}_{\text{cs}}[Y]$ and let $(\rho^Y, \mathcal{H}^Y)$ be a representation such that $\|S\|_{\max} = \|\rho(S)\|$. Now $\mathcal{H}^X = \mathbb{C}_{\text{cs}}[X] \otimes \mathcal{H}^Y$ is a representation of $\mathbb{C}_{\text{cs}}[X]$ by Lemma 4.8.2. Moreover, the restriction of $\rho^X(\beta(S))$ to $\mathcal{H}^Y = A\mathcal{H}^X$ equals $\rho^Y(S)$, so $\|\beta(S)\|_{\max} \geq \|\rho^X(\beta(S))\| \geq \|\rho^Y(S)\|_{\max} = \|S\|_{\max}$.

Conversely, let $(\rho^X, \mathcal{H}^X)$ be a representation of $\mathbb{C}_{\text{cs}}[X]$ such that $\|\beta(S)\|_{\max} = \|\rho^X(\beta(S))\|$. We have $\mathcal{H}^X = A\mathcal{H}^X + (1-A)\mathcal{H}^X$ where $\mathcal{H}^Y = A\mathcal{H}^X$ is a representation of $\mathbb{C}_{\text{cs}}[Y]$. The restriction of $\rho^X(\beta(S))$ to $\mathcal{H}^Y$ equals $\rho^Y(S)$ and the restriction of $\rho^X(\beta(S))$ to $(1-A)\mathcal{H}^X$ equals zero, because $\beta(S) \in A\mathbb{C}_{\text{cs}}[X]A$. So $\|S\|_{\max} \geq \|\rho^Y(S)\| = \|\rho^X(\beta(S))\| = \|\beta(S)\|_{\max}$.

Hence $\beta\colon \mathbb{C}_{\text{cs}}[Y] \to A\mathbb{C}_{\text{cs}}[X]A$ is an isometry for the maximal norms, and it extends to an isometric embedding $\beta\colon C^*_{\max}(Y) \hookrightarrow C^*_{\max}(X)$. Its image is the closure of $A\mathbb{C}_{\text{cs}}[X]A$, this is $AC^*_{\max}(X)A$. □

Lemma 4.8.4. *We have $C^*_{\max}(X)A\ker(\Phi^X)A = I_c(X)A$ and $\beta(I_c(Y)) = AI_c(X)A$.*

Proof. Recall that $I_c(X) = C^*_{\max}(X)\ker(\Phi^X)$, so clearly $C^*_{\max}(X)A\ker(\Phi^X)A \subseteq I_c(X)A$. To show the converse, it is enough to show that $\ker(\Phi^X)A \subseteq \mathbb{C}_{\text{cs}}[X]A\ker(\Phi^X)A$. Let $T \in \ker(\Phi^X)A$. For every $y, y' \in Y$ the map $\pi_{U_y}Ti_{U_{y'}}\colon L^2U_{y'} \to L^2U_y$ does not change when it is composed with $i_{U_{y'}}Ai_{U_{y'}}\colon L^2U_{y'} \to L^2U_{y'}$, so it has rank at most 1. There is an integer N, such that for every $y \in Y$ there are at most N different $y' \in Y$ such that $\pi_{U_y}Ti_{U_{y'}} \neq 0$, because T has controlled support. Therefore $\operatorname{im}(T) \cap L^2U_y$ has rank at most N. Write $\operatorname{im}(T) \cap L^2U_y = f_{1y}\mathbb{C} \oplus \cdots \oplus f_{Ny}\mathbb{C}$ where the f_{ky} are pairwise perpendicular elements of L^2X (some of them may be zero). For all $1 \leq k \leq N$ and $y \in Y$ let $B_{ky}\colon L^2U_y \to L^2U_y$ be a rank-one isometry sending $f_{ky}\mathbb{C}$ to $\mathbb{1}_{U_y}\mathbb{C}$ (or let $B_{ky} = 0$ if $f_{ky} = 0$). Then $B^*_{ky}AB_{ky} = B^*_{ky}B_{ky}$ is the projection on $f_{ky}\mathbb{C}$. Then $\sum_{k=1}^N B^*_{ky}AB_{ky}$ is the projection on $\operatorname{im}(T) \cap L^2U_y$. Now define $B_k = \sum_y B_{ky}$. This is a bounded operator with controlled support so we have $B_k \in \mathbb{C}_{\text{cs}}[X]$. Moreover $T = \sum_y \sum_{k=1}^N B^*_{ky}AB_{ky} = \sum_{k=1}^N B^*_kAB_kT \in \mathbb{C}_{\text{cs}}[X]A\ker(\Phi^X)A$.

For the second claim note that for any $S \in \mathbb{C}_{\text{cs}}[Y]$ we have $\Phi^Y(S) = 0$ if and only if $\Phi^X(\beta(S)) = 0$. Thus $\beta(I_c(Y)) = \beta(C^*_{\max}(Y)\ker(\Phi^Y)) = AC^*_{\max}(X)A\ker(\Phi^X)A = AI_c(X)A$. □

Now we can prove Lemma 4.8.1.

Proof of Lemma 4.8.1. We will use the characterisation of geometric property (T) given in Proposition 4.7.10.

Suppose that (X, μ) has geometric property (T). Let $p \in C^*_{\max}(X)$ be a projection generating the left ideal $I_c(X)$. Since $1 - A \in I_c(X)$, it is a multiple of p, and from this it follows that ApA is again a projection. Then we see that $ApA \in AI_c(X)A$ and for all $t \in AI_c(X)A$ we have

$tApA = tA = t$, so ApA generates $AI_c(X)A$ as a left $AC^*_{\max}(X)A$-module. So, by Lemma 4.8.4 we know that $I_c(Y)$ is generated by a projection, hence (Y,ν) has geometric property (T).

Conversely, suppose that (Y,ν) has geometric property (T). Let $q \in C^*_{\max}(Y)$ be a projection generating the left ideal $I_c(Y)$. By Lemma 4.8.4 we have $\beta(q) \in AI_c(X)A$. In particular, $\beta(q)(1-A) = 0$, and from this it follows that $\beta(q)+1-A$ is again a projection. We have

$$\begin{aligned}
&C^*_{\max}(X)(\beta(q)+1-A)\\
=&C^*_{\max}(X)\beta(q)+C^*_{\max}(X)(1-A)\\
=&C^*_{\max}(X)AI_c(X)A+C^*_{\max}(X)(1-A)\\
=&I_c(X)A+C^*_{\max}(X)(1-A)\\
=&I_c(X)
\end{aligned}$$

by Lemma 4.8.4, so (X,μ) has geometric property (T). □

Now we show that geometric property (T) is independent of the measure for uniformly locally finite spaces.

Lemma 4.8.5. *Let Y be a uniformly locally finite space and let ν be a measure on Y such that all points are measurable and have measure uniformly bounded from above and below. Denote the counting measure on Y by κ. Then (Y,ν) has property (T) if and only if (Y,κ) has property (T).*

Proof. Define $N \in L^\infty(Y)$ by $N(y) = \nu(y)^{\frac{1}{2}}$. We will consider this as a multiplication operator in $\mathbb{C}_{\mathrm{cs}}[Y]$. Consider the linear map $a\colon L^2(Y,\nu) \to L^2(Y,\kappa)$ given by $a(\eta) = N\eta$. This is a unitary map and it induces an isomorphism $\mathbb{C}_{\mathrm{cs}}[Y,\kappa] \to \mathbb{C}_{\mathrm{cs}}[Y,\nu]$ sending T to a^*Ta. For any $y \in Y$ and large enough $V \subseteq Y$, we have

$$\Phi_\nu(a^*Ta)(y) = a^*Ta(\mathbb{1}_V)(y) = N^{-1}TN(\mathbb{1}_V)(y) = \Phi_\kappa(N^{-1}TN)(y),$$

so $\Phi_\nu(a^*Ta) = \Phi_\kappa(N^{-1}TN)$.

We now identify the algebras $\mathbb{C}_{\mathrm{cs}}[Y,\kappa] \cong \mathbb{C}_{\mathrm{cs}}[Y,\nu] \cong \mathbb{C}_{\mathrm{cs}}[Y]$, just remembering that $\Phi_\nu(T) = \Phi_\kappa(N^{-1}TN)$. We get $C^*_{\max}(Y,\nu) = C^*_{\max}(Y,\kappa)$. Moreover we have $\ker(\Phi_\nu) = N\ker(\Phi_\kappa)N^{-1}$ and therefore $I_c(Y,\nu) = NI_c(Y,\kappa)N^{-1}$. We see that $I_c(Y,\nu)$ is generated by a projection if and only if $I_c(Y,\kappa)$ is generated by a projection. By Proposition 4.7.10 we are done. □

Finally we prove that property (T) does not depend on the chosen measure and is also a coarse invariant.

Theorem 4.A. Suppose (X,μ) and (X',μ') are coarsely equivalent spaces with bounded geometry, equipped with uniformly bounded measures for which gordo sets exist. Then (X,μ) has property (T) if and only if (X',μ') has property (T). In particular, whether X has property (T) is independent of the chosen uniformly bounded measure μ for which a gordo set exists.

Proof. Construct Y and Y' as before. Then Y, X, X', Y' are all coarsely equivalent. Since Y and Y' are coarsely equivalent there are coarsely dense $Z \subseteq Y$ and $Z' \subseteq Y'$ and a coarse isomorphism $f\colon Z \to Z'$ (see Proposition 2.1.12). Taking the counting measure on Z and Z', it is easy to see that f induces an isomorphism between the algebras $\mathbb{C}_{\mathrm{cs}}[Z]$ and $\mathbb{C}_{\mathrm{cs}}[Z']$ commuting with Φ^Z and $\Phi^{Z'}$, so that Z has property (T) if and only if Z' has property (T).

We can write Y as a bounded disjoint union $\bigcup_{z\in Z} V_z$ such that $z \in V_z$. Define the measure λ on Z by $\lambda(z) = \nu(V_z)$. Similarly define a measure λ' on Z'. Now by Lemmas 4.8.1 and 4.8.5 the following are equivalent:

– The space (X, μ) has property (T).

– The space (Y, ν) has property (T).

– The space (Z, λ) has property (T).

– The space Z has property (T) with the counting measure.

– The space Z' has property (T) with the counting measure

– The space (Z', λ') has property (T).

– The space (Y', ν') has property (T).

– The space (X', μ') has property (T).

$\square$

4.9 Amenability

In this section, we will consider which connected coarse spaces X have geometric property (T). Recall that a coarse space X is connected if $\{(x, y)\}$ is a controlled set for all $x, y \in X$. As in [58], it turns out that an unbounded connected coarse space X has geometric property (T) precisely if it does not satisfy a version of amenability. First, we give two equivalent definitions of amenability for connected coarse spaces. It is a generalisation of amenability for uniformly locally finite metric spaces, as defined e.g. in [58].

Proposition 4.9.1. *Let X be a connected coarse space and μ a uniformly bounded measure for which a gordo set exists. The following are equivalent:*

(i) There is a positive unital linear map $\varphi\colon L^\infty X \to \mathbb{C}$ such that if (A_i) is a blocking collection and $f \in L^\infty X$ satisfies $\int_{A_i} f = 0$ for all i, and f is zero outside of the union $\bigsqcup_i A_i$, then $\varphi(f) = 0$.

(ii) For each controlled measurable set $E \subseteq X \times X$ and each $\varepsilon > 0$ there is a non-empty bounded measurable $U \subseteq X$ such that $\mu(E_U) \leq (1 + \varepsilon)\mu(U)$.

In this case, we say that X is amenable.

Proof. First assume (ii). Define the directed set

$$\Lambda = \{(E, \varepsilon) \mid E \subseteq X \times X \text{ symmetric, measurable, controlled and containing the diagonal, } \varepsilon > 0\}$$

with $(E, \varepsilon) \geq (E', \varepsilon')$ if $E \supseteq E'$ and $\varepsilon \leq \varepsilon'$. For each $\lambda = (E, \varepsilon) \in \Lambda$, choose a measurable set $U_\lambda \subseteq X$ such that $\mu(E_{U_\lambda}) \leq (1 + \varepsilon)\mu(U_\lambda)$. Define $\varphi_\lambda \in (L^\infty X)^*$ by $\varphi_\lambda(f) = \frac{1}{\mu(U_\lambda)} \int_{U_\lambda} f$. Then the φ_λ form a net in the unit ball of $(L^\infty X)^*$. By the Banach-Alaoglu theorem there is a limit point φ.

We will now show that φ satisfies the condition. So let (A_i) be a blocking collection and $f \in L^\infty X$ such that $\int_{A_i} f = 0$ for all i and f is zero outside $\bigsqcup_i A_i$. Let E be the symmetric measurable controlled set $\sqcup_i A_i \times A_i \cup \{(x, x) \mid x \in X\}$. Let $\lambda = (E', \varepsilon)$ with $E' \supseteq E$. Since E_{U_λ} is a union of some of the A_i and points outside of $\bigsqcup_i A_i$, we have $\int_{E_{U_\lambda}} f = 0$. Then $\varphi_\lambda(f) = \frac{1}{\mu(U_\lambda)} \int_{U_\lambda} f =$

$-\frac{1}{\mu(U_\lambda)}\int_{E_{U_\lambda}\setminus U_\lambda} f$ and $|\varphi_\lambda(f)| \leq \frac{\mu(E_{U_\lambda}\setminus U_\lambda)}{\mu(U_\lambda)} \cdot \|f\|_\infty \leq \varepsilon\|f\|_\infty$. This shows that $\varphi_\lambda(f) \to 0$ as $\lambda \to \infty$, hence $\varphi(f) = 0$.

Conversely, assume (i). Let $\varphi\colon L^\infty X \to \mathbb{C}$ be a positive unital linear map satisfying the condition. Let E be a gordo set and assume for contradiction that there is $\varepsilon > 0$ such that for all bounded measurable $U \subseteq X$ we have $\mu(E_U) > (1+\varepsilon)\mu(U)$. Let (A_i) be a blocking collection with union X, such that each A_i has measure at least $\varepsilon' > 0$. We may assume without loss of generality that $E = \{A_i \times A_j \mid (A_i \times A_j) \cap E \neq \emptyset\}$. Because of bounded geometry, we can use a colouring argument to find involutions $\sigma_1, \dots, \sigma_N$ on the index set I such that $E_{A_i} = A_{\sigma_1(i)} \cup \dots \cup A_{\sigma_N(i)}$ for all $i \in I$. For $1 \leq k \leq N$ let $E_k = \bigsqcup_i A_i \times A_{\sigma_k(i)}$. Note that Δ^{E_k} can be viewed as an operator on L^1 and also as an operator on $L^\infty X$. For any $f \in L^\infty X$ we have $\int_{A_i \cup A_{\sigma_k(i)}} \Delta^{E_k} f = 0$ for all i, so by the condition on φ we have $\varphi(\Delta^{E_k} f) = 0$. Equivalently, $(\Delta^{E_k})^*\varphi = 0$.

Let $\mathcal{P}(X) \subseteq L^1 X$ denote the positive measurable functions on X with integral 1. We can view $L^1 X$ as a subset of $(L^\infty X)^*$ using the standard pairing between $L^1 X$ and $L^\infty X$. By the Goldstine theorem, φ is in the weak closure of $\mathcal{P}(X)$. Note that $(\Delta^{E_k})^*$ is a weakly continuous function on $(L^\infty X)^*$ and on $L^1 X$ it is the same as Δ^{E_k}. Therefore, 0 is in the weak closure of the convex set

$$\{(\Delta^{E_k}\psi) \mid \psi \in \mathcal{P}(X)\} \subseteq \bigoplus_{k=1}^{N} L^1(X).$$

Since norm-closed convex sets are also weakly closed, it follows that 0 is also in the norm closure of this set. So we can choose $\psi \in \mathcal{P}(X)$ such that all $\Delta^{E_k}\psi$ are small in the L^1-norm. In particular, we can choose $\psi \in \mathcal{P}(X)$ such that $\sum_i M_i \leq \eta$, where $M_i = \sup_k \int_{A_i} |\Delta^{E_k}\psi|$ and we will choose the constant $\eta > 0$ later.

Let $\delta = \frac{1}{1+\frac{1}{2}\varepsilon} < 1$. For $n \in \mathbb{Z}$ define $U_n = \bigsqcup\{A_i \mid \int_{A_i} \psi \geq \delta^n \mu(A_i)\}$. Since $\int_X \psi = 1$, the U_n have bounded measure, they consist of finitely many of the A_i and they are bounded. Moreover, $U_n = \emptyset$ for n small enough. By assumption, we have $\mu(E_{U_n}) \geq (1+\varepsilon)\mu(U_n)$ for each n. For each i, define n_i as the minimal number such that $A_i \subseteq E_{U_{n_i}}$ and N_i as the maximal number such that $A_i \not\subseteq U_{N_i}$ (these may be ∞). Then $\int_{A_i} \psi < \delta^{N_i}\mu(A_i)$, and there is k such that $\int_{A_{\sigma_k(i)}} \psi \geq \delta^{n_i}\mu(A_{\sigma_k(i)})$. Now

$$M_i \geq \int_{A_i} |\Delta^{E_k}\psi| = \int_{A_i} \left| \mu(A_{\sigma_k(i)})\psi - \int_{A_{\sigma_k(i)}} \psi \right| \geq \varepsilon'(\delta^{n_i} - \delta^{N_i}).$$

Define

$$D_n = E_{U_n} \setminus U_{n+1} = \bigsqcup\{A_i \mid n_i \leq n \leq N_i - 1\}.$$

Then

$$\sum_n \delta^n \mu(D_n) = \sum_i \mu(A_i) \sum_{n=n_i}^{N_i-1} \delta^n = \frac{1}{1-\delta}\sum_i \mu(A_i)(\delta^{n_i} - \delta^{N_i}) \leq \frac{1}{(1-\delta)\varepsilon'}\sum_i M_i \leq \frac{\eta}{(1-\delta)\varepsilon'}.$$

For all n we have

$$(1+\varepsilon)\mu(U_n) \leq \mu(E_{U_n}) \leq \mu(U_{n+1}) + \mu(D_n).$$

Multiplying this inequality by δ^n on both sides and adding it for all n we get

$$(1+\varepsilon)\sum_n \delta^n \mu(U_n) \leq \frac{1}{\delta}\sum_n \delta^n \mu(U_n) + \sum_n \delta^n \mu(D_n),$$

so

$$(1+\varepsilon-\frac{1}{\delta})\sum_n \delta^n\mu(U_n)\leq \frac{\eta}{(1-\delta)\varepsilon'}$$

and

$$\sum_n \delta^n\mu(U_n)\leq \frac{2\eta}{(1-\delta)\varepsilon\varepsilon'}.$$

On the other hand, we have

$$1=\int\psi=\sum_n\int_{U_{n+1}\setminus U_n}\psi\leq\sum_n\delta^n(\mu(U_{n+1})-\mu(U_n))=(\frac{1}{\delta}-1)\sum_n\delta^n\mu(U_n).$$

Combining these inequalities gives $1\leq\frac{2\eta}{\delta\varepsilon\varepsilon'}$. Picking $\eta=\frac{\delta\varepsilon\varepsilon'}{4}$ gives the desired contradiction. □

The following proposition is a generalisation of [58, Proposition 6.1]. It gives several characterisations of amenability of a coarse space X in terms of its algebra.

Proposition 4.9.2. *Let X be a connected coarse space and μ a uniformly bounded measure for which a gordo set exists. The following are equivalent:*

(i) The space X is amenable.

(ii) There is a net ξ_λ of unit vectors in L^2X satisfying $\left\|T\xi_\lambda-M_{\Phi(T)}\xi_\lambda\right\|\to 0$ for all $T\in\mathbb{C}_{\mathrm{cs}}[X]$ for which $\Phi(T)$ is bounded.

(iii) For all $T\in\mathbb{C}_{\mathrm{cs}}[X]$ with $\Phi(T)=0$, we have $0\in\sigma(T)$.

(iv) For all gordo sets E, we have $0\in\sigma_{\max}(\Delta^E)$.

(v) There is a unital representation $\mathcal{H}$ of $\mathbb{C}_{\mathrm{cs}}[X]$ that has a non-zero constant vector.

(vi) There is a positive unital linear map $\varphi\colon L^\infty X\to\mathbb{C}$ satisfying $\varphi(Tf)=\varphi(\overline{\Phi(T^*)}\cdot f)$ for all $T\in\mathbb{C}_{\mathrm{cs}}[X]$ for which $\varphi(T^*)$ is bounded.

Proof. First, assume that X is amenable. We will use criterion (ii) from Proposition 4.9.1. As in the proof of Proposition 4.9.1, define the directed set

$$\Lambda=\{(E,\varepsilon)\mid E\subseteq X\times X\text{ gordo},\varepsilon>0\}$$

with $(E,\varepsilon)\geq(E',\varepsilon')$ if $E\supseteq E'$ and $\varepsilon\leq\varepsilon'$. For each $\lambda=(E,\varepsilon)\in\Lambda$, choose a bounded measurable subset $U_\lambda\subseteq X$ such that $\mu((E\circ E)_{U_\lambda})\leq(1+\varepsilon)\mu(U_\lambda)$. Define $\xi_\lambda=\mu(U_\lambda)^{-\frac{1}{2}}\mathbb{1}_{U_\lambda}$. This is a unit vector in L^2X.

Let $T\in\mathbb{C}_{\mathrm{cs}}[X]$ such that $\Phi(T)$ is bounded. We will show that $\left\|T\xi_\lambda-M_{\Phi(T)}\xi_\lambda\right\|\to 0$. We can assume without loss of generality that $\Phi(T)=0$. Let T be supported on E, let $E'\supseteq E$ and $\varepsilon>0$. Then for $\lambda=(E',\varepsilon)$, the function ξ_λ is constant on $E_{X\setminus U_\lambda}$, so $T\xi_\lambda$ is supported on E_{U_λ}. Since $T\mathbb{1}_{(E\circ E)_{U_\lambda}}$ is zero on E_{U_λ}, we know that $T\xi_\lambda$ is the restriction to E_{U_λ} of $-T\eta_\lambda$, where $\eta_\lambda=\mu(U_\lambda)^{-\frac{1}{2}}\mathbb{1}_{(E\circ E)_{U_\lambda}\setminus U_\lambda}$. Now

$$\|T\xi_\lambda\|\leq\|T\eta_\lambda\|\leq\|T\|\cdot\|\eta_\lambda\|=\|T\|\,\mu(U_\lambda)^{-\frac{1}{2}}\mu((E\circ E)_{U_\lambda}\setminus U_\lambda)^{\frac{1}{2}}\leq\|T\|\cdot\varepsilon^{\frac{1}{2}},$$

so indeed $T\xi_\lambda \to 0$.

The implications (ii) $\implies$ (iii) $\implies$ (iv) are trivial.

Suppose (iv) is true. Let Λ be the set

$$\{(E,\varepsilon) \mid E \subseteq X \times X \text{ measurable controlled}, \varepsilon > 0\}.$$

We equip this with the partial order $(E,\varepsilon) \geq (E',\varepsilon')$ if $E \supseteq E'$ and $\varepsilon \leq \varepsilon'$. Then Λ is a directed set. Let $\mathcal{U}$ be an ultrafilter on $\mathcal{P}(\Lambda)$ containing $\{\lambda' \geq \lambda\}$ for each $\lambda \in \Lambda$. By assumption, for every $\lambda = (E,\varepsilon) \in \Lambda$ there is a representation $\mathcal{H}_\lambda$ of $\mathbb{C}_{\mathrm{cs}}[X]$ and a unit vector $v_\lambda \in \mathcal{H}_\lambda$ satisfying $\left\|\Delta^E v_\lambda\right\| \leq \varepsilon$. Note that for $\lambda' \geq \lambda$ we also have $\left\|\Delta^E v_{\lambda'}\right\| \leq \varepsilon$. Now let $\mathcal{H} = \prod_{\lambda\in\Lambda} \mathcal{H}_\lambda/\mathcal{U}$ be the ultraproduct of the $\mathcal{H}_\lambda$ and let $v = \lim_{\mathcal{U}} v_\lambda \in \mathcal{H}$. Then v is a unit vector and for each measurable controlled set E we have $\Delta^E v = \lim_{\mathcal{U}} \Delta^E v_\lambda = 0$. By Lemma 4.7.4, v is a constant vector, showing (iv) $\implies$ (v).

For (v) $\implies$ (vi), let $v \in \mathcal{H}$ be a constant vector of norm 1 and define $\varphi\colon L^\infty X \to \mathbb{C}$ by $\varphi(f) = \langle M_f v, v\rangle$. Then φ is a positive unital linear map and for all $f \in L^\infty X$ and $T \in \mathbb{C}_{\mathrm{cs}}[X]$ such that $\Phi(T^*)$ is bounded, we have

$$\varphi(Tf) = \langle M_{\Phi(TM_f)}v, v\rangle = \langle TM_f v, v\rangle = \langle M_f v, M_{\Phi(T^*)}v\rangle = \langle M_{\overline{\Phi(T^*)}\cdot f}v, v\rangle = \varphi(\overline{\Phi(T^*)}\cdot f).$$

Finally, for (vi) $\implies$ (i), let $\varphi\colon L^\infty X \to \mathbb{C}$ be a positive unital linear map satisfying $\varphi(Tf) = \varphi(\overline{\Phi(T^*)}\cdot f)$ for all $T \in \mathbb{C}_{\mathrm{cs}}[X]$ such that $\Phi(T^*)$ is bounded. Let (A_i) be a blocking collection and $E = \bigsqcup_i A_i \times A_i$. Let $f \in L^\infty X$ be a function such that $\int_{A_i} f = 0$ for all i, and f is zero outside of $\bigsqcup_i A_i$. Then $\Delta^E f = f$ and $\varphi(\Delta^E) = 0$, so $\varphi(f) = \varphi(\Delta^E f) = 0$. Therefore, X is amenable (using part (i) of Proposition 4.9.1).

$\square$

Remark 4.9.3. If E' is as in proposition 4.7.9, the properties in Theorem 4.9.2 are also equivalent to $0 \in \sigma_{\max}(\Delta^{E'})$.

Now we can give the generalisation of [58, Corollary 6.1].

Theorem 4.B. Let X be a connected coarse space and let μ be a uniformly bounded measure for which a gordo set exists. If X is bounded, then it is amenable and has geometric property (T). If X is unbounded, it has geometric property (T) if and only if it is not amenable.

Proof. If X is bounded, we have already shown that it has geometric property (T), and it is amenable because $\mathbb{1}_X \in L^2X$ is a constant vector in the standard representation.

Suppose that X is unbounded and not amenable. Then there is a gordo set E such that $0 \notin \sigma_{\max}(\Delta^E)$. Then for all representations $(\rho,\mathcal{H})$ of $\mathbb{C}_{\mathrm{cs}}[X]$ we have $\ker(\rho(\Delta^E)) = \mathcal{H}_c = \emptyset$ and Δ^E has spectral gap. By proposition 4.7.6 the space X has geometric property (T).

Suppose that X is unbounded, has geometric property (T) and is amenable. By Proposition 4.7.6 there is a measurable controlled set E such that for all representations $(\rho,\mathcal{H})$ of $\mathbb{C}_{\mathrm{cs}}[X]$, we have $\mathcal{H}_c = \ker(\rho(\Delta^E))$, and Δ^E has spectral gap. Since X is amenable, we have $0 \in \sigma(\Delta^E)$, but then 0 is an isolated point in the spectrum of $\Delta^E \in B(L^2X)$. So there is a non-zero $\xi \in \ker(\Delta^E)$, which is then a constant vector. Since $\xi \in L^2X$ and X is unbounded, ξ can not be constant as a function. So there are measurable sets $A, B \subseteq X$ of positive measure such that the convex hulls of $\xi(A)$ and $\xi(B)$ are disjoint. If we intersect A with all of the members of a blocking collection, one of the intersections must have positive measure, since blocking collections are countable. Therefore, we can assume that A is bounded, and similarly, that B is bounded. Since X is connected, $A \cup B$ is also bounded. Let $T \in \mathbb{C}_{\mathrm{cs}}[X]$ be the operator defined by

$$T\eta(x) = \begin{cases} \frac{1}{\mu(B)} \int_B \eta - \frac{1}{\mu(A)} \int_A \eta & \text{if } x \in A; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Phi(T) = 0$ but $T\xi \neq 0$, hence ξ is not a constant vector and we have a contradiction. ☐

Corollary 4.9.4. *Amenability for connected coarse spaces of bounded geometry does not depend on the measure chosen and is a coarse invariant.*

Proof. This follows from Theorem 4.A and Theorem 4.B. It is also straightforward to show it directly, using criterion (ii) of Proposition 4.9.1. ☐

4.10 Manifolds

Let (M, g) be an n-dimensional complete Riemannian manifold, not necessarily connected. Consider the geodesic distance d and volume μ. The metric d defines a coarse structure on M, where $E \subseteq M \times M$ is controlled if $\sup_{(x,y)\in E} d(x, y) < \infty$. For any $R > 0$ let E_R be the controlled set $\{(x, y) \in M \times M \mid d(x, y) \leq R\}$. Let $B(x, R) = (E_R)_x$ denote the ball of radius R around a point $x \in M$. In [20], it is defined that M has *bounded geometry* if the injectivity radius is positive, and the Ricci curvature is bounded below (by a possibly negative constant). In this section, we assume that this is the case.

By Bishop's inequality (see [17, Theorem 3.101.i]) there are constants $B, B' > 0$ such that $\mu(B(x, R)) \leq B \exp(B'R)$ for every $x \in M$ and $R > 0$. Hence, the measure μ is uniformly bounded. There are also $R, A > 0$ such that $\mu(B(x, R)) \geq A$ for all $x \in M$, by [9, Proposition 14] (the Proposition is stated for compact manifolds, but the proof works just as well in the non-compact case if the injectivity radius is positive). In particular, the controlled set E_R is gordo, so we can conclude that the manifold M has bounded geometry as a coarse space.

Let Δ_M be the (positive) Laplacian operator on M and let $\Delta_H = 1 - \exp(-\Delta_M)$. Since the spectrum of Δ_M is contained in $[0, \infty]$, the spectrum of Δ_H is contained in $[0, 1]$. In particular, Δ_H is a bounded operator. We can write

$$\Delta_H f(x) = f(x) - \int_M p(x, y) f(y) d\mu(y)$$

where $p(x, y)$ is the heat kernel on M. We need some estimates on this function.

Lemma 4.10.1. *(i) There are constants $c, R > 0$ such that $p(x, y) \geq c$ whenever $d(x, y) \leq R$.*

(ii) The manifold is stochastically complete*: for every $x \in M$ we have $\int_M p(x, y) d\mu(y) = 1$.*

(iii) For every $\varepsilon > 0$ there is an R such that $\int_{B(x,R)} p(x, y) d\mu(y) \geq 1 - \varepsilon$ for all $x \in M$.

Proof. (i) This follows from [20, Equation 7.43].

(ii) This follows from [20, Theorem 3.4] and the bound on $\mu(B(x, R))$ given above.

(iii) The conditions of [20, Equation 6.40] are met, so there are constants $C, D > 0$ such that

$$p(x,y) \le C \exp\left(\frac{-d(x,y)^2}{D}\right)$$

for all $x, y \in M$. Now for any $x \in M$ and $R > 0$ we have

$$\begin{aligned}\int_{M\setminus B(x,R)} p(x,y)d\mu(y) &\le C\int_{M\setminus B(x,R)} \exp\left(\frac{-d(x,y)^2}{D}\right) d\mu(y)\\ &\le C\int_R^\infty \frac{2r}{D}\exp\left(\frac{-r^2}{D}\right) V_r(x)dr\\ &\le \frac{2BC}{D}\int_R^\infty r^{n+1}\exp\left(\frac{-r^2}{D}\right) dr.\end{aligned}$$

As this last integral converges, we know that for every $\varepsilon > 0$ there is an $R > 0$ such that $\int_{M\setminus B(x,R)} p(x,y)d\mu(y) \le \varepsilon$. Then by part (ii) we get $\int_{B_{x,R}} p(x,y)d\mu(y) \ge 1-\varepsilon$ for all $x \in M$. ☐

Lemma 4.10.2. *We have $\Delta_H \in C^*_{\max}(M)$.*

Proof. For any $R > 0$ let $\Delta_{HR} \in C_{\mathrm{cs}}[M]$ be the operator defined by

$$\Delta_{HR}f(x) = \int_{B(x,R)} p(x,y)(f(x)-f(y))d\mu(y).$$

Then for $R' > R$ we have

$$(\Delta_{HR'} - \Delta_{HR})f(x) = \int_{B(x,R')\setminus B(x,R)} p(x,y)(f(x)-f(y))d\mu(y).$$

For large enough R we have $\int_{M\setminus B(x,R)} p(x,y)d\mu(y) \le \varepsilon$. Then we have

$$\langle(\Delta_{HR'} - \Delta_{HR})\xi,\xi\rangle = \frac{1}{2}\int_{\{(x,y)\in M^2 | R<d(x,y)\le R'\}} p(x,y)|\xi(x)-\xi(y)|^2 d(\mu\times\mu)(x,y) \le 2\varepsilon\|\xi\|^2.$$

So

$$0 \le_{\max} \Delta_{HR'} - \Delta_{HR} \le_{\max} 2\varepsilon.$$

Therefore the Δ_{HR} form a Cauchy sequence in $\mathbb{C}_{\mathrm{cs}}[M]$ with the maximal norm, and they converge to Δ_H. ☐

For any $R > 0$ let $\Delta_R = \Delta^{E_R}$. We will use the estimates of the heat kernel from Lemma 4.10.1 to prove estimates on Δ_H in terms of the Δ_R.

Lemma 4.10.3. *We have the following estimates on Δ_H:*

(i) There are constants $c, R > 0$ such that $\Delta_H \ge_{\max} c\Delta_R$.

(ii) For every $\varepsilon > 0$ there are constants $d, R > 0$ such that $\Delta_H \leq_{\max} \varepsilon + d\Delta_R$.

Proof. (i) Take the same constants c, R as in Lemma 4.10.1 part (i). Then

$$(\Delta_H - c\Delta_R)f(x) = \int_M \alpha(x,y)(f(x) - f(y))d\mu(y),$$

where $\alpha(x,y) = p(x,y)$ if $d(x,y) > R$ and $\alpha(x,y) = p(x,y) - c$ if $d(x,y) \leq R$. Since $\alpha \geq 0$ everywhere we have $\Delta_H \geq_{\max} c\Delta_R$.

(ii) Let Δ_{HR} be as in the proof of Lemma 4.10.2. There is a large enough R such that $\Delta_H \leq_{\max} \varepsilon + \Delta_{HR}$. The function $p(x,y)$ is bounded from above by some constant d, and then we get $\Delta_{HR} \leq_{\max} d\Delta_R$.

□

For any representation $\mathcal{H}$ of $C^*_{\max}(M)$, we can consider the unbounded symmetric operator $\Delta_M = -\log(1 - \Delta_H)$ on $\mathcal{H}$. We say that Δ_M has spectral gap if there is a constant $\gamma > 0$ such that the spectrum of this operator is contained in $\{0\} \cup [\gamma, \infty)$ for each representation $\mathcal{H}$.

Theorem 4.C. Let (M, g) be a Riemannian manifold of bounded geometry equipped with the coarse structure coming from the geodesic metric. Then M has geometric property (T) if and only if the Laplacian Δ_M has spectral gap in $C^*_{\max}(M)$.

Proof. For any $R > 0$, the controlled set E_R generates the coarse structure on M and is gordo. So we can apply Proposition 4.7.9 to see that M has geometric property (T) if and only if Δ_R has spectral gap. Clearly, Δ_H has spectral gap if and only if Δ_M has spectral gap. For the remainder of the proof, we use the estimates of Lemma 4.10.3.

Suppose that M has geometric property (T). By Lemma 4.10.1 part (i), there are constants $c, R > 0$ such that $\Delta_H \geq c\Delta_R$. Now Δ_R has spectral gap, and so does Δ_H, and also Δ_M.

Suppose that Δ_H has spectral gap. There is $\varepsilon > 0$ such that $\sigma_{\max}(\Delta_H) \subseteq \{0\} \cup [2\varepsilon, \infty)$. By Lemma 4.10.1 part (ii) there are $b, R > 0$ such that $\Delta_H \leq \varepsilon + b\Delta_R$. Then Δ_R has spectral gap, so M has geometric property (T). □

4.11 Warped systems

Let Γ be a finitely generated group with finite generating set S. Let (M, g) be a compact Riemannian manifold. Let d be the distance function on M and μ the measure on M defined by the metric g. Let $\alpha \colon \Gamma \curvearrowright M$ be an action by diffeomorphisms. Let $\mathcal{C}M = M \times \{1, 2, \dots\}$ be the discrete cone. This is equipped with the Riemannian metric $g_{\mathcal{C}}$ that is $t \cdot g$ on $M_t = M \times \{t\}$. Let $d_{\mathcal{C}}$ be the corresponding distance function, given by $d_{\mathcal{C}}(x,y) = t \cdot d(x,y)$ when $x, y \in M_t = M \times \{t\}$ and $d_{\mathcal{C}}(x,y) = \infty$ if x, y are in different M_t. Now define the *warped system* $\mathrm{Warp}(\Gamma \curvearrowright M)$ as the space $\mathcal{C}M$ equipped with the largest metric δ_Γ for which $\delta_\Gamma(x,y) \leq d_{\mathcal{C}}(x,y)$ and $\delta_\Gamma(x, s \cdot x) \leq 1$ for all $s \in \Gamma$ and $x, y \in \mathcal{C}M$. We will consider $\mathrm{Warp}(\Gamma \curvearrowright M)$ as a coarse space.

In [53], it is shown that that $X = \mathrm{Warp}(\Gamma \curvearrowright M)$ does not depend on the generating set of Γ as a coarse space. It is also coarsely equivalent to the coarse disjoint union $\bigsqcup_{t \in \mathbb{Z}} M \times \{t\}$ of subspaces of the warped cone as introduced by Roe in [45] (see [53, Lemma 6.5]).

Let μ be the measure on X defined through the Riemannian metric $g_{\mathcal{C}}$. Since the manifold M is compact, the injectivity radius is positive and the Ricci curvature is bounded below. It follows as in the previous section that $F_R = \{(x,y) \in X \times X \mid d_{\mathcal{C}}(x,y) \leq R\}$ is gordo for any $R > 0$, and that its balls are uniformly bounded. By Lemma 4.11.7 below, it follows that μ is uniformly bounded. In particular, X has bounded geometry.

Let $\Delta_\Gamma = \sum_{s\in S} 1 - s \in \mathbb{C}[\Gamma]$. This can also be viewed as an element in $\mathbb{C}_{\mathrm{cs}}[X]$. In [53], it is shown that a uniformly locally finite, coarsely dense subspace of X is a family of expanders if and only if there is a constant $\gamma > 0$ such that $\langle \Delta_\Gamma \xi, \xi\rangle \geq \gamma \|\xi\|^2$ for all $\xi \in L^2 M$ with $\int_M \xi d\mu = 0$. We want to consider when these graphs have geometric property (T), or equivalently by Theorem 4.A, when X has geometric property (T).

Definition 4.11.1. Let Γ be a group and (M,μ) a measure space. Let $\alpha\colon \Gamma \curvearrowright M$ be a measurable action. The action is called *ergodic* if the only measurable Γ-invariant subsets of M are the empty set and M itself.

If Γ has property (T) and the action on M is ergodic, we might expect the warped system X to have geometric property (T). At the moment, this is an open question.

Question 4.11.2. *Suppose Γ has property (T) and the action $\alpha\colon \Gamma \curvearrowright M$ is ergodic. Does X have geometric property (T)?*

The measure-preserving action of Γ on M gives a map $\Gamma \to U(L^2X)$ given by $g\cdot\xi(x) = \xi(g^{-1}x)$ for $g \in \Gamma, \xi \in L^2X$. The image is contained in $\mathbb{C}_{\mathrm{cs}}[X]$. Then the map $\Gamma \to \mathbb{C}_{\mathrm{cs}}[X]$ gives rise to a map $\mathbb{C}[\Gamma] \to \mathbb{C}_{\mathrm{cs}}[X]$. Now every representation of $\mathbb{C}_{\mathrm{cs}}[X]$ pulls back to a representation of $\mathbb{C}[\Gamma]$, therefore the map $\mathbb{C}[\Gamma] \to \mathbb{C}_{\mathrm{cs}}[X]$ is contracting for the maximal norms. So it induces a map $C^*_{\max}(\Gamma) \to C^*_{\max}(X)$. If Γ has property (T), then we know that Δ_Γ has spectral gap in $C^*_{\max}(\Gamma)$. So the image of Δ_Γ also has spectral gap in $C^*_{\max}(X)$. If $\ker(\rho(\Delta)) = \mathcal{H}_c$ for any representation $(\rho,\mathcal{H})$ of $\mathbb{C}_{\mathrm{cs}}[X]$, then we can conclude that X has property (T) by Proposition 4.7.6. We can prove the following partial result.

Lemma 4.11.3. *Suppose that Γ has property (T) and the action $\alpha\colon \Gamma \curvearrowright M$ is ergodic. Let $\mathcal{H}$ be a representation of $\mathbb{C}_{\mathrm{cs}}[X]$, and $v \in \mathcal{H}$ such that $\Delta_\Gamma v = 0$. For any $f \in L^\infty X$ such that $\int_{X_t} f d\mu = 0$ for all $t \in \mathbb{N}$, we have $\langle M_f v, v\rangle = 0$.*

Proof. We can assume without loss of generality that v is a unit vector.

The composition $\Gamma \xrightarrow{\alpha} \mathbb{C}_{\mathrm{cs}}[X] \xrightarrow{\rho} B(\mathcal{H})$ gives an action of Γ on $\mathcal{H}$, still denoted by α. For any $s \in S$, we have $\|(1-\alpha_s)v\|^2 = \langle (1-\alpha_s^*)(1-\alpha_s)v, v\rangle = \langle (2-\alpha_s-\alpha_s^*)v, v\rangle$. Since $2-\alpha_s-\alpha_s^* \leq \Delta_\Gamma$ it follows that $(1-\alpha_s)v = 0$. So v is Γ-invariant.

Now define $\varphi \in (L^\infty X)^*$ by $\varphi(f) = \langle M_f v, v\rangle$. Note that φ is a mean on $L^\infty X$. Moreover, φ is Γ-invariant: for any $g \in \Gamma, f \in L^\infty X$ we have

$$\varphi(\alpha_g f) = \langle M_{\alpha_g f} v, v\rangle = \langle \alpha_g M_f \alpha_g^* v\rangle = \langle M_f v, v\rangle = \varphi(f).$$

Let $\mathcal{P}(X) = \{\psi \in L^1X \mid \psi \geq 0, \int_X \psi d\mu = 1\}$. By the Goldstine theorem, φ is in the weak closure of $\mathcal{P}(X)$. Let $f \in L^\infty X$ such that $\int_{X_t} f d\mu = 0$ for all $t \in \mathbb{N}$. Let $\varepsilon > 0$. Then 0 is in the weak closure of the convex set

$$\left\{ ((1-\alpha_s)\psi) \in \sum_{s\in S} L^1 X \mid \psi \in \mathcal{P}(X), |\psi(f) - \varphi(f)| < \varepsilon \right\}.$$

Since norm-closed convex sets are also weakly closed, 0 is also in the norm closure of the set above. Hence there is $\psi \in \mathcal{P}(X)$ with $|\psi(f) - \varphi(f)| < \varepsilon$ and $\|(1-\alpha_s)\psi\|_1 < \varepsilon$ for all $s \in S$. Now let

$\xi = \psi^{\frac{1}{2}} \in L^2X$. This is a unit vector satisfying

$$\left\|(1-\alpha_s)\xi\right\|_2^2 = \int_X (\xi(s\cdot x) - \xi(x))^2 d\mu(x) \leq \int_X |\xi(s\cdot x)^2 - \xi(x)^2| = \left\|(1-\alpha_s)\psi\right\|_1 < \varepsilon$$

for all $s \in S$. Then $\|\Delta_\Gamma \xi\|_2 \leq \varepsilon^{\frac{1}{2}}|S|$. Let $\xi_c \in L^2X$ be the locally constant function defined by $\xi_c(x) = \frac{1}{\mu(X_t)}\int_{X_t}\xi$ for $x \in X_t$. There is $\gamma > 0$ such that $\sigma(\Delta_\Gamma) \subseteq \{0\} \cup [\gamma, \infty)$. Now $\xi - \xi_c$ is perpendicular to the locally constant functions, and the locally constant functions are the only Γ-invariant functions in L^2X, since the action of Γ is ergodic. From $\left\|\Delta_\Gamma(\xi - \xi_c)\right\|_2 \leq \varepsilon^{\frac{1}{2}}|S|$ it now follows that $\left\|(\xi-\xi_c)\right\|_2 \leq \gamma^{-1}\varepsilon^{\frac{1}{2}}|S|$. Now

$$\begin{aligned}
|\varphi(f)| &\leq |\psi(f)| + \varepsilon \\
&= \left|\int_X \xi(x)^2 f(x) d\mu(x)\right| + \varepsilon \\
&= \left|\int_X (\xi(x) - \xi_c(x))^2 f(x) d\mu(x)\right| + 2\left|\int_X (\xi(x) - \xi_c(x))\xi_c(x) f(x) d\mu(x)\right| + \varepsilon \\
&\leq \gamma^{-2}\varepsilon|S|^2 \cdot \|f\|_\infty + \gamma^{-1}\varepsilon^{\frac{1}{2}}|S| \cdot \|f\|_\infty + \varepsilon.
\end{aligned}$$

Since this holds for any $\varepsilon > 0$, we conclude that $\varphi(f) = 0$. □

Remark 4.11.4. In the above, we only used that $\sigma(\Delta_\Gamma) \subseteq \{0\} \cup [\gamma, \infty)$ instead of the stronger condition $\sigma_{\max}(\Delta_\Gamma) \subseteq \{0\} \cup [\gamma, \infty)$.

Remark 4.11.5. To see why the Lemma is progress to the Question, consider that we have to prove that every $v \in \ker(\rho(\Delta_\Gamma))$ is constant. For this we can assume that $v \in \mathcal{H}_c^\perp$, and try to show that it is in fact 0. Now from the Lemma it follows that in this case, we have $v \in (\rho(L^\infty X)\mathcal{H}_c)^\perp$, in other words, v is a vector that is in some way quite far from being constant.

To see this, consider $f \in L^\infty X$ and $w \in \mathcal{H}_c$. We can write $f = f_c + g$ with f_c constant on each component of X and $\int_{X_t} g d\mu = 0$ for all g. For all $t \in \mathbb{C}$ we have $v + tw \in \ker(\rho(\Delta_\Gamma))$, so by the Lemma, we have $\langle M_g(v+tw), v+tw\rangle = 0$. Using some different values of t gives $\langle v, M_g w\rangle = 0$. Also $\langle v, M_{f_c} w\rangle = 0$ since $M_{f_c} w$ is a constant vector. So $\langle v, M_f w\rangle = 0$.

The converse is also an open question.

Question 4.11.6. *Suppose the action α is free and X has geometric property (T). Is it necessary that Γ has property (T) and that the action is ergodic?*

We have some partial results on this question. Before we state them, we give some preliminary results.

Lemma 4.11.7. *For $R > 0$ and $g \in \Gamma$ define the controlled sets $E_R = \{(x,y) \in X \mid \delta_\Gamma(x,y) \leq R\}$, $E_g = \{(g\cdot x, x) \mid x \in X\}$ and $F_R = \{(x,y) \in X \mid d_{\mathcal{C}}(x,y) \leq R\}$. For any $R > 0$ there is $R' > 0$ such that $E_R \subseteq \bigcup_{g \in S^{\lfloor R \rfloor}} F_{R'} \circ E_g$.*

Proof. For any $s \in S$ the map $M \to M,\ x \to s\cdot x$ is Lipschitz. So there is a constant $L > 0$ such that $d_{\mathcal{C}}(s\cdot x, s\cdot y) \leq L \cdot d_{\mathcal{C}}(x,y)$. Let $R' = L^R R$. For any $(y,x) \in E_R$, there are $x_0 = x, x_1, \ldots, x_n \in X$ and $y_0, y_1, \ldots, y_n = y \in X$ and $s_1, s_2, \ldots, s_n \in S$ such that $s_k y_{k-1} = x_k$ for $k = 1, 2, \ldots, n$ and

$$n + d_{\mathcal{C}}(x_0, y_0) + \ldots + d_{\mathcal{C}}(x_n, y_n) \leq R.$$

It follows by induction that

$$d_{\mathcal{C}}(s_k \cdots s_2 \cdot s_1 x_0, y_k) \leq L^k d_{\mathcal{C}}(x_0, y_0) + \ldots + L d_{\mathcal{C}}(x_{k-1}, y_{k-1}) + d_{\mathcal{C}}(x_k, y_k).$$

In particular

$$d_{\mathcal{C}}(g \cdot x, y) \leq L^n d_{\mathcal{C}}(x_0, y_0) + \ldots + L d_{\mathcal{C}}(x_{n-1}, y_{n-1}) + d_{\mathcal{C}}(x_n, y_n) \leq L^R R = R'$$

for $g = s_n \cdots s_2 \cdot s_1$. Now $(g \cdot x, x) \in E_g$ and $(y, g \cdot x) \in F_{R'}$ so $(y, x) \in F_{R'} \circ E_g$. Since $g \in S^{\lfloor R \rfloor}$ this shows that

$$E_R \subseteq \bigcup_{g \in S^{\lfloor R \rfloor}} F_{R'} \circ E_g.$$

□

Note that $I = \bigoplus_{t \in \mathbb{N}} B(L^2 X_t)$ is a two-sided ideal in $\mathbb{C}_{\mathrm{cs}}[X]$.

Lemma 4.11.8. *Suppose that the action α is free. Then $\mathbb{C}_{\mathrm{cs}}[X]/I$ is naturally isomorphic to the algebraic cross product $\mathbb{C}_{\mathrm{cs}}[\mathcal{C}M]/I \rtimes \Gamma$.*

Proof. The natural map $\mathbb{C}_{\mathrm{cs}}[\mathcal{C}M] \rtimes \Gamma \to \mathbb{C}_{\mathrm{cs}}[X]$ descends to $\mathbb{C}_{\mathrm{cs}}[\mathcal{C}M]/I \rtimes \Gamma \to \mathbb{C}_{\mathrm{cs}}[X]/I$.

To show that it is surjective, let $T \in \mathbb{C}_{\mathrm{cs}}[X]$. Then T is supported on E_R for some integer $R > 0$. By Lemma 4.11.7, there is $R' > 0$ such that $E_R \subseteq \bigcup_{g \in S^R} F_{R'} E_g$. We can use this decomposition to write T as a finite sum $T = \sum_g A_g \alpha_g$, with $A_g \in \mathbb{C}_{\mathrm{cs}}[\mathcal{C}M]$ showing that the map is indeed surjective.

For injectivity, suppose that $\sum_{g \in B} A_g \alpha_g \in I$, with $B \subset \Gamma$ finite and $A_g \in \mathbb{C}_{\mathrm{cs}}[\mathcal{C}M]$. Then we will show that $A_g \in I$ for all $g \in B$. We may as well assume that $\sum_{g \in B} A_g \alpha_g = 0$. Since the action is free and by homeomorphisms, for any $g \in \Gamma$ we have $\min_{x \in M} d(g \cdot x, x) > 0$. There is some $R > 0$ such that all A_g are supported on F_R. Now let $T = \frac{2R}{\min_{g \neq h \in B} \min_{x \in M} d(gh^{-1} \cdot x, x)}$. For any $g \in B$ we know that $A_g \alpha_g$ is supported on $F_R E_g$. We also have $A_g \alpha_g = -\sum_{h \in B \setminus \{g\}} A_h \alpha_h$, so $A_g \alpha_g$ is supported on $F_R E_g \cap \bigcup_{h \in B \setminus \{g\}} F_R E_h$. Suppose that $(x, y) \in F_R E_g \cap F_R E_h$. Then $d_{\mathcal{C}}(x, g \cdot y) \leq R$ and $d_{\mathcal{C}}(x, h \cdot y) \leq R$, so $d_{\mathcal{C}}(g \cdot y, h \cdot y) \leq 2R$. For some t we have $x, y \in X_t$ and then $d_{\mathcal{C}}(g \cdot y, h \cdot y) \geq t \cdot \min_{z \in M} d(gh^{-1} \cdot z, z) \geq \frac{2Rt}{T}$. So $t \leq T$. Hence we have $F_R E_g \cap F_R E_h \subseteq \bigcup_{t \leq T} X_t \times X_t$, and $F_R E_g \cap \bigcup_{h \in B \setminus \{g\}} F_R E_h \subseteq \bigcup_{t \leq T} X_t \times X_t$. So $A_g \alpha_g$ is supported on $\bigcup_{t \leq T} X_t \times X_t$, and we conclude that $A_g \in \bigoplus_{t \in \mathbb{N}} B(L^2 X_t)$. □

Lemma 4.11.9. *Suppose the action α is free. Then the natural map $C^*_{\max}(\Gamma) \to C^*_{\max}(X)$ is an injection.*

Proof. The proof is similar to the proof of [58, Lemma 7.1]. Let $\overline{I}$ denote the closure of I in $C^*_{\max}(X)$. The previous lemma gives $\mathbb{C}_{\mathrm{cs}}[X]/I \cong \mathbb{C}_{\mathrm{cs}}[\mathcal{C}M]/I \rtimes \Gamma$, hence $C^*_{\max}(X)/\overline{I} \cong C^*_{\max}(\mathcal{C}M)/\overline{I} \rtimes_{\max} \Gamma$. Now it suffices to prove that the composition

$$C^*_{\max}(\Gamma) \to C^*_{\max}(X) \to C^*_{\max}(X)/\overline{I} \cong C^*_{\max}(\mathcal{C}M)/\overline{I} \rtimes_{\max} \Gamma$$

is injective.

For any $t \in \mathbb{N}$ define the map $\varphi_t \colon \mathbb{C}_{\mathrm{cs}}[\mathcal{C}M] \to \mathbb{C}$ by $\varphi_t(T) = \frac{1}{\mu(X_t)} \langle \mathbb{1}_{X_t}, T \mathbb{1}_{X_t} \rangle$. This extends to a positive unital map $\varphi_t \colon C^*_{\max}(\mathcal{C}M) \to \mathbb{C}$. Let φ be a limit point of these maps. Then $\varphi \colon C^*_{\max}(\mathcal{C}M)/\overline{I} \to \mathbb{C}$ is a ucp map. Since maximal crossed products are functorial for ucp maps, this gives a map $C^*_{\max}(\mathcal{C}M)/\overline{I} \rtimes_{\max} \Gamma \to C^*_{\max}(\Gamma)$. Now the composition $C^*_{\max}(\Gamma) \to C^*_{\max}(\mathcal{C}M)/\overline{I} \rtimes_{\max} \Gamma \to C^*_{\max}(\Gamma)$ is the identity, so the first map is an injection. □

As a result, we get that for free actions, Δ_Γ has spectral gap in $C^*_{\max}(X)$ if and only if Γ has property (T). So if Γ does not have property (T), for any $\varepsilon > 0$, there is some representation $(\rho, \mathcal{H})$ and a unit vector $v \in \mathcal{H}_c^\perp$ with $\|\Delta_\Gamma v\| < \varepsilon$. Suppose that $\|\Delta_R v\| < 1$ for some $0 < R < 1$. Then one can show that $\|\Delta_R v\| \to 0$ as $R \to 0$. Let $u\colon L^2 X \to L^2 X$ be given by $u(\xi)(x,t) = \xi(x, 2t)$. Precomposing the representation ρ by the map $\tau\colon \mathbb{C}_{\mathrm{cs}}[X] \to \mathbb{C}_{\mathrm{cs}}[X], \tau(T) = uTu^*$ multiple times gives vectors $v_0 = v, v_1, \dots$ such that $\|\Delta_\Gamma v_n\| < \varepsilon$ and $\|\Delta_R v_n\| < \varepsilon$. In this case it follows that X does not have geometric property (T). A similar strategy would work for non-ergodic actions. The main obstacle is, that there may be vectors v for which $\|\Delta_R v\| = 1$ for all $0 < R < 1$.

5 Fixed point properties for bornological groups

This chapter is based on the preprint [51], which is joint work with Romain Tessera. Large parts of it are reproduced verbatim.

Chapter summary

We investigate fixed point properties for isometric actions of topological groups on a wide class of metric spaces, with a particular emphasis on Hilbert spaces. Instead of requiring the action to be continuous, we assume that it is "controlled", i.e. compatible with respect to some natural left-invariant coarse structure. For locally compact groups, we prove that these coarse fixed point properties are equivalent to the usual ones, defined for continuous actions. We deduce generalisations of two results of Gromov originally stated for discrete groups. For Polish groups with bounded geometry (in the sense of Rosendal), we prove a version of Serre's theorem on the stability of coarse property FH under central extensions. As an application we prove that the group $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ has property FH. Finally, we characterise geometric property (T) for sequences of finite Cayley graphs in terms of coarse property FH of a certain group.

5.1 Introduction and main results

In this chapter we investigate rigidity properties of *bornological* groups (see Section 2.1.3). There is a natural notion of *controlled* isometric action of a bornological group on a metric space (see Definition 5.1.2). Given a class $\mathcal{X}$ of metric spaces, we say that a bornological group G has *coarse* property F$\mathcal{X}$ if every controlled action of G on a space from $\mathcal{X}$ admits a fixed point. For a topological group G (which we always assume to be Hausdorff), we shall say that G has *topological* property F$\mathcal{X}$ if every *continuous* isometric action of G on any space in $\mathcal{X}$ has a fixed point.

We shall work with metric spaces satisfying a certain convexity property: *uniformly convex* metric spaces (see Definition 5.2.1). This is motivated by the fact that such spaces admit a natural notion of centre for bounded sets (see Lemma 5.2.2). This class encompasses a wide family of complete geodesic metric spaces such as uniformly convex (real) Banach spaces (in particular L^p-spaces for $1 < p < \infty$) and CAT(0) spaces. We recall that by the Mazur-Ulam theorem, isometric actions on Banach spaces are affine.

This introduction is organised as follows.

- First, in §5.1.1, we introduce the bornology $\mathcal{OB}$ associated to a topological group (G, τ). We also define a coarse version of Kazhdan's property (T), and we state the relevant version of the Delorme-Guichardet theorem relating coarse property (T) and coarse property FH.

- In §5.1.2, we state our main result regarding locally compact groups, which roughly says that topological and coarse fixed point properties coincide. We use it to extend two theorems, that were originally proved by Gromov for discrete groups, to locally compact groups.

- In §5.1.3, we consider bornological groups with *bounded geometry*. Besides locally compact groups this class contains interesting examples of Polish groups studied by Rosendal [46]. We prove an analogue of Serre's theorem on the stability of property (T) under central extensions for bornological groups with bounded geometry. We apply this to prove that certain Polish groups which are quasi-isometric to $\mathbb{Z}$, such as $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$, have coarse (and therefore topological) property FH.

– In §5.1.4, we consider a topological group naturally associated to a sequence of Cayley graphs of finite groups. We show that *topological* and *coarse* property (T) of this group encode *expansion* and *geometric property (T)* (in the sense of Willett and Yu) respectively.

5.1.1 Bornological groups

Recall from Example 2.1.16 that any topological group (G, τ) has an associated *pre-compact bornology* $\mathcal{K}$, consisting of all pre-compact subsets of G. There is another natural bornology on G.

Definition 5.1.1. (i) Let $\sigma : G \curvearrowright X$ be an isometric action of a group G on a metric space X. Then $\mathcal{B}_\sigma$ consisting of all subsets A such that $\sigma(A) \cdot x$ has finite diameter for all $x \in X$ is a bornology.

(ii) Let (G, τ) be a topological group. We can also make it a bornological group with the bornology $\mathcal{OB} = \cap_\sigma \mathcal{B}_\sigma$, where σ runs over all continuous isometric actions on metric spaces.

This bornology has been introduced in [46] (see Definition 5.3.3 for more details). For sigma-compact locally compact groups, $\mathcal{K} = \mathcal{OB}$.

Definition 5.1.2. Let $(G, \mathcal{B})$ be a bornological group. An isometric action $\sigma : G \curvearrowright X$ is called controlled if $\mathcal{B} \subseteq \mathcal{B}_\sigma$, i.e. for $A \in \mathcal{B}$, $\sigma(A)$ has bounded orbits. A bornological group has coarse property FX if all its controlled isometric actions on X have a fixed point.

Note that a norm-preserving representation on a Banach space is automatically controlled (since orbits are obviously bounded).

We now proceed to define a convenient notion of property (T) for bornological groups. We recall that all unitary representations are controlled, so we will not impose any restriction on them. However, we need to specify the meaning of "almost" invariant vectors. For simplicity, we focus on Hilbert spaces in the introduction (see Definition 5.3.11 for a more general treatment).

Definition 5.1.3. Let G be a bornological group. A representation $\pi \colon G \to U(\mathcal{H})$ has *almost invariant vectors* if for each $\varepsilon > 0$ and bounded $A \subseteq G$, there is a unit vector $v \in \mathcal{H}$ such that $\sup_{a \in A} \lVert \pi(a)v - v \rVert < \varepsilon$.

Definition 5.1.4. A bornological group G has *coarse property (T)* if every representation with almost invariant vectors has an invariant vector.

Adapting the proof of Delorme and Guichardet's famous theorem, we prove that coarse property (T) implies coarse property FH, and the converse is true if the bornology is generated by a single bounded set (see Theorem 5.5.1).

5.1.2 The case of locally compact groups

The following result says that under a mild assumption, when studying isometric actions of topological groups on uniformly convex metric spaces, one can restrict to continuous ones.

Proposition 5.1.5 (see §5.3.1)**.** *Let G be a topological group and let U be an open neighbourhood of the identity. Let X be a uniformly convex metric space. Let $\alpha \colon G \curvearrowright X$ be an isometric action. Let $c > 0$ and suppose there is $x \in X$ such that $d(\alpha(g)x, x) \le c$ for all $g \in U$. Then there is $y \in X$ with $d(x, y) \le 2c$, such that the restriction of α to the closed convex hull of $\alpha(G)y$ is continuous.*

If X is a Banach space, then the closed convex hull of $\alpha(G)y$ can be replaced by the closed vector subspace spanned by $\alpha(G)y$ in the proposition. We refer to the discussion in §5.7 for a cohomological interpretation of Proposition 5.1.5 in the case where X is a Banach space.

We give two different proofs of this result, both based on certain notions of centres (see §5.2). One of them only applies to actions of locally compact groups on uniformly convex Banach spaces. However, this one exploits a new notion of centre of a finitely additive measure, which we believe could be useful for future applications.

Proposition 5.1.5 has the following consequence, formulated in terms of bornological groups.

Theorem 5.A (See Corollary 5.3.14)**.** Let X be a uniformly convex metric space. Equipped with the precompact bornology, a locally compact group G has topological property FX if and only if it has coarse property FX.

Note that coarse property FX is obviously stronger than topological property FX, hence Theorem 5.A is really about the other implication. Similar statements can be derived for properties defined in terms of norm-preserving representations (see Corollary 5.3.13).

We now present two consequences, which are new in general for locally compact groups, but are due to Gromov [21, 3.8.B] for discrete groups. Before stating them, we will make the following **standing assumption** on a family $\mathcal{X}$ of metric spaces:
(*) They must consist of uniformly convex spaces and be stable under the following natural operations: ultraproducts; rescaling of the metric; taking closed geodesically convex subspaces (resp. closed vector subspaces when working with Banach spaces).

The main examples one can have in mind are the following classes:

(i) the class of Hilbert spaces;

(ii) the class of closed subspaces of L^p-spaces, for a fixed $1 < p < \infty$;

(iii) more generally: any class of uniformly convex Banach spaces which is stable under ultraproducts;

(iv) the class of CAT(0) spaces;

(v) the class of $\mathbb{R}$-trees.

Theorem 5.B (See Theorem 5.4.12)**.** Let $\mathcal{X}$ be a family of uniformly convex metric spaces, closed under taking rescaled ultraproducts and closed under taking closed convex subspaces. Let G be a compactly generated locally compact group such that every continuous isometric action on an element of $\mathcal{X}$ has almost fixed points. Then G has property F$\mathcal{X}$.

This theorem was proved by Shalom [48, Theorem 6.1] for isometric actions of locally compact groups on Hilbert spaces (Shalom states it in terms of reduced cohomology).

Theorem 5.C (See Theorem 5.4.15)**.** Let $\mathcal{X}$ be a family of uniformly convex metric spaces, closed under taking rescaled ultraproducts and closed under taking closed convex subspaces. Let G be a compactly generated locally compact group with property F$\mathcal{X}$. Then there exists a compactly presented locally compact group G' with property F$\mathcal{X}$, and a continuous surjective homomorphism $G' \to G$ with discrete kernel.

Two remarks are in order:

– Theorems 5.B and 5.C have natural versions for bornological groups: see Corollary 5.4.5 and Theorem 5.4.7.

– Theorem 5.C is a special case of a more general result saying that property F$\mathcal{X}$ is open in the relevant *marked group topology* for locally compact compactly generated groups (see Theorem 5.4.13).

As a special case, we recover the following known result.

Corollary 5.1.6. *Let G be a locally compact group with property (T). Then there exists a compactly presented locally compact group G' with property (T), and a continuous surjective homomorphism $G' \to G$ with discrete kernel.*

Corollary 5.1.6 has been proved by Shalom [48, Theorem 6.7] for discrete groups. It has later been proved by Fisher and Margulis for locally compact groups [15, Theorem 2.4.]. Recently, de la Salle proved a version of Theorem 5.C for isometric actions of locally compact groups on certain families of Banach spaces [47, Corollary 5.13.]. Like ours, the previous proofs are based on an ultralimit argument. A difficulty comes from the fact that when defined naïvely, an ultralimit of isometric actions may not be continuous. However, it remains easy to ensure that orbits of compact sets are bounded, so in our case, we simply can apply Theorem 5.A to replace it by a continuous one. The main interest of our proof is that it applies to isometric actions on non-linear geodesic metric spaces.

5.1.3 Bornological groups with bounded geometry

We now consider a property of bornological groups which can be seen a coarse analogue of being locally compact for topological groups.

Definition 5.1.7. A bornological group G has *bounded geometry* if there is a bounded set C such that all bounded sets A can be covered by finitely many left translates of C. Such a C is called *gordo* if it is also symmetric.

Remark 5.1.8. It is easy to see that G has bounded geometry precisely if it has bounded geometry as a coarse space with the induced left-invariant coarse structure (cf. Definition 4.3.3).

Remark 5.1.9. If C is gordo, then all bounded sets A can also be covered by finitely many right translates of C, for if $A^{-1} \subseteq g_1C \cup \cdots \cup g_kC$ then $A \subseteq C^{-1}g_1^{-1} \cup \cdots \cup C^{-1}g_k^{-1}$.

Remark 5.1.10. If a bornological group G is such that its bornology is generated by a bounded set S, then its bounded structure coincides with $\mathcal{B}_\sigma$ where σ is the action of G on its Cayley graph $\mathrm{Cay}(G, S)$. In other words, its coarse structure is the one associated with the left-invariant word metric associated to S. Moreover G has bounded geometry if and only if S^n is contained in a union of finitely many left-translates of S for all $n \geq 1$.

Equipped with the bornology $\mathcal{K}$, a locally compact group has bounded geometry: a gordo set is given by any symmetric compact neighbourhood of the identity. More generally, any subgroup H of a locally compact group G, equipped with the induced bornology from G, has bounded geometry.

A more interesting source of examples are Polish groups with bounded geometry in the sense of Rosendal [46, Chapter 5]. Lots of explicit examples can be obtained from [38, Theorem 3], and we will see one of them shortly.

Recall that by a well-known theorem of Serre, if we have short exact sequence of locally compact groups

$$1 \to C \to G \to Q \to 1,$$

such that Q has (T), and C is central and contained in $\overline{[G,G]}$, then G has (T) [5, Theorem 1.7.11]. A more general version of this result holds for property F$\mathcal{E}$ where $\mathcal{E}$ is any class of uniformly convex Banach space which is stable under ultraproduct [11, Theorem 6.3.]. We do not know whether [11, Theorem 6.3.] holds for all bornological groups. However, we are able to prove a version of Serre's theorem for bornological groups with bounded geometry. For A, B subsets of a coarse group G, we say that A is *coarsely contained* in B if there is a bounded $C \subseteq G$ with $A \subseteq C \cdot B$.

Theorem 5.D (See Theorem 5.5.7)**.** Let $(G, \mathcal{B})$ be a bornological group with bounded geometry, whose bornology is generated by a single set. Let $p\colon G \to Q$ be a surjective morphism and equip Q with the induced bornology $p(\mathcal{B})$. Assume that the kernel $A = \ker(p)$ is central in G and coarsely contained in the commutator $[G,G]$. Then G has coarse property (T) if and only if Q has coarse property (T).

Note that even for locally compact groups equipped with $\mathcal{K}$, the condition on A is weaker than the one in Serre's theorem, so in particular we recover it.

To showcase the power of Theorem 5.D, we will use it to study certain Polish groups considered in [46]. First, let $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ be the group of all lifts of orientation-preserving homeomorphisms h of the circle S^1, equipped with the compact open topology. It is one explicit example of a Polish group with bounded geometry that can be obtained from [38, Theorem 3]. Equipped with $\mathcal{OB}$, this Polish group is coarsely equivalent to $\mathbb{Z}$, and hence has bounded geometry (see [46, §3.4.]).

Theorem 5.E (See Corollary 5.5.8)**.** The group $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ equipped with $\mathcal{OB}$ has coarse property (T). Hence, it also has coarse property FH (and in particular topological property FH).

We get similar results for groups such as the group $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ of lifts of order-preserving permutations of $\mathbb{Q}/\mathbb{Z}$ and the group $AC^*_{\mathbb{Z}}(\mathbb{R})$ of lifts of absolutely continuous orientation-preserving homeomorphisms of the circle whose inverses are absolutely continuous.

We turn to an important step in the proof of Theorem 5.D. Let us say that an isometric action of a bornological group is *well controlled* if the orbits of bounded sets are relatively compact. Note that this notion is more demanding than being controlled. In particular it is non-empty for linear actions: if G is a locally compact non-discrete group, then the regular representation of G as a discrete group (on $\ell^2(G)$) is never well controlled.

Definition 5.1.11. A bornological group G has *coarse property (T-)* if every well-controlled representation with almost invariant vectors has an invariant vector. It has *coarse property FH-* if every well-controlled isometric action on a Hilbert space has a fixed point.

As for coarse property (T), we prove the relevant version of Delorme-Guichardet's theorem (See Theorem 5.5.1).

Clearly coarse property (T) implies coarse property (T-). Since both properties are stronger than usual property (T) for locally compact groups, we deduce from Theorem 5.A that they are equivalent for locally compact groups. It turns out that this equivalence can be extended to bornological groups with bounded geometry.

Theorem 5.F (See Theorem 5.5.4)**.** For bornological groups with bounded geometry, coarse (T) and coarse (T-) are equivalent. Similarly, coarse FH and coarse FH- are equivalent for these groups.

Proposition 5.1.12 (See Proposition 5.5.3)**.** *Let G be a bornological group with bounded geometry and let $\alpha\colon G \curvearrowright \mathcal{H}$ be a controlled affine action on a Hilbert space. There is a non-empty invariant closed subspace $\mathcal{H}'$ such that the restriction of α to $\mathcal{H}'$ is well controlled.*

They are also equivalent for the bounded products appearing in Theorem 5.H. In fact, we have no example where they are not equivalent, although we suspect they exist.

Theorem 5.F is a consequence of the following result of independent interest, which is in the same spirit as Proposition 5.1.5.

Contrary to Proposition 5.1.5, which has a version for actions on very general geodesic metric spaces, we do not know whether this one holds beyond actions on Hilbert spaces. This comes from the method of proof: Proposition 5.1.5 exploits a notion of "centre" of bounded subsets which makes sense in a wide class of metric spaces. By contrast, the proof of Proposition 5.1.12 is based on a more subtle notion of centre: shopping centres, which we could only define in Hilbert spaces.

5.1.4 Bounded products

We now turn to a class of topological groups for which topological (T) and coarse (T) are distinct (and similarly for property FH). Let Γ_n be a sequence of (discrete) groups, and let S_n be a symmetric set of generators of Γ_n. For each $x \in \Gamma_n$ we define the length $l(x)$ to be the least natural number such that $x \in S_n^{l(x)}$.

Definition 5.1.13. The *bounded product* of a sequence of groups Γ_n with symmetric generating sets S_n is the group

$$\prod_n \{\Gamma_n, S_n\} = \{(g_n) \in \prod_n \Gamma_n \mid \sup_n l(g_n) < \infty\}.$$

Let $G = \prod_n \{\Gamma_n, S_n\}$. This has a canonical bornology $\mathcal{B}$, where $A \subseteq G$ is bounded if and only if $\sup_{(g_n)\in A} \sup_n l(g_n) < \infty$.

There is also a natural topology on G: for any proper function $\rho\colon \mathbb{N} \to \mathbb{Z}_{\geq 0}$, define

$$U_\rho = \{(g_n) \in G \mid l(g_n) \leq \rho(n)\}.$$

The sets U_ρ, together with their translates, form a basis for the topology on G.

Proposition 5.1.14 (See Proposition 5.6.1)**.** *With respect to this topology, we have $\mathcal{B} = \mathcal{K} = \mathcal{OB}$, provided that all S_n are finite.*

We shall say that a topological group G has *topological* property (T) if all continuous unitary representations with almost invariant vectors (relative to compact subsets) have invariant vectors (see Definition 5.3.4).

If the Γ_n are all finite groups and $|S_n|$ is bounded, we can characterise exactly when G has coarse or topological property (T).

Theorem 5.G (See Theorem 5.6.2)**.** Let Γ_n be a sequence of finite groups with symmetric generator sets S_n. Suppose that $\sup_n |S_n| < \infty$. Let X be the coarse disjoint union of the Cayley graphs $\mathrm{Cay}(\Gamma_n, S_n)$. Then the bounded product G has topological property (T) if and only if X is an expander sequence.

Given a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ there is an equivalence relation $\sim$ on G, with $(g_n) \sim (h_n)$ if $\{n \mid g_n = h_n\} \in \mathcal{U}$. Taking the quotient by this equivalence relation gives a group $G_{\mathcal{U}} = \prod_n \{\Gamma_n, S_n\}/\mathcal{U}$, generated by the image of $\prod_n S_n$. We will view it as a discrete group. If $|S_n| = k$ is a fixed integer, then the groups obtained as $G_{\mathcal{U}}$ have a natural interpretation in terms of the space of marked groups, i.e. quotients of the free groups on k generators: indeed, it consists of the closure of $\{(\Gamma, S_n) \mid n \in \mathbb{N}\}$.

Theorem 5.H (See Theorem 5.6.6)**.** Let Γ_n be a sequence of finite groups with symmetric generator sets S_n. Suppose that $\sup_n |S_n| < \infty$. Let X be the coarse disjoint union of the Cayley graphs $\mathrm{Cay}(\Gamma_n, S_n)$. The following are equivalent:

(i) The coarse space X has geometric property (T).

(ii) All ultraproducts $G_{\mathcal{U}} = \prod_n \Gamma_n/\mathcal{U}$ have property (T) with uniform spectral gap.

(iii) All ultraproducts $G_{\mathcal{U}}$ have property (T).

(iv) There is a finitely generated group Γ with property (T) such that all Γ_n are a quotient of Γ.

(v) The bounded product G has coarse property (T).

(vi) The bounded product G has coarse property (T-).

The equivalence between (i) and (ii) are proved in [37, Theorem 1]. To the best of our knowledge the equivalence between (i) and (iv) is new, but is reminiscent of a similar statement proved in [57]: if $\Gamma_n = \Gamma/\Lambda_n$, where Λ_n is a decreasing sequence of finite index normal subgroups of Γ such that $\cap_n \Lambda_n = \{1\}$, then the coarse space X has geometric property (T) if and only if Γ has (T).

5.2 Centres in uniformly convex metric spaces

Let E be a Banach space. It is *reflexive* if the natural map to the double dual $E \to E^{**}$ is an isomorphism. It is called superreflexive if all its ultrapowers are reflexive. A Banach space is superreflexive if and only if it has an equivalent norm that is uniformly convex, i.e. for every $\varepsilon > 0$ there is $\delta > 0$ such that if x, y are unit vectors with $\|x - y\| \geq \varepsilon$, then $\left\| \frac{x+y}{2} \right\| < 1 - \delta$. Moreover, for any isometric action σ of a group G on a superreflexive space X, there exists an equivalent uniformly convex norm $\|\cdot\|'$ on X such that σ is isometric on $(X, \|\cdot\|')$, see [14]. Examples of superreflexive Banach spaces are Hilbert spaces and all L^p spaces with $1 < p < \infty$.

Even more generally, we can consider a uniquely geodesic complete metric space X. For any $x, y \in X$, the geodesic between them is denoted by $[x, y]$ and the midpoint by m_{xy}. The distance between x and y is denoted by $|xy|$. A subset $A \subseteq X$ is *convex* if $x, y \in A$ implies $[x, y] \subseteq A$. For any $A \subseteq X$, the *convex hull* $\mathrm{co}(A)$ is the smallest convex set that contains it. Its closure is denoted by $\overline{\mathrm{co}}(A)$.

We take the following notion of uniform convexity from [39, Definition 7.1].

Definition 5.2.1. A uniquely geodesic complete metric space X is *uniformly convex* if for each $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y, z \in X$ with $|xy| \geq \varepsilon \max(|xz|, |yz|)$, we have $|zm_{xy}| \leq (1 - \delta) \max(|xz|, |yz|)$.

We will say "uniformly convex metric space" as short-hand for "uniformly convex uniquely geodesic complete metric space". This generalises uniform convexity for Banach spaces. As a matter of notation, we will write δ_ε for a δ that works.

5.2.1 Centres of bounded sets

Let X be a uniformly convex metric space. Let $A \subseteq X$ be a non-empty bounded subset. For $x \in X$ let $r(x) = \sup_{a \in A} |ax|$. Let $\rho(A) = \inf_{x \in X} r(x)$. As shown in [39, Proposition 7.2], there is a unique point $Z(A)$, called the centre, that satisfies $r(Z(A)) = \rho(A)$. There are no additional regularity requirements for A. It is an isometric invariant, meaning that whenever $u \in \mathrm{Aut}(X)$ is an isometry, we have $Z(uA) = uZ(A)$. In this section we describe this centre and some of its variants, which will be the key ingredient in many of our proofs.

For convenience we provide the construction of the centre.

Lemma 5.2.2. *Let $A \subseteq X$ be a non-empty bounded subset of a uniformly convex metric space. There is a unique point $Z(A)$ satisfying $r(Z(A)) = \rho(A)$.*

Proof. Let $x_n \in X$ be a sequence of points satisfying $\lim_{n\to\infty} r(x_n) = \rho(A)$. We will show that it is a Cauchy sequence. Let $\varepsilon > 0$ and let m, n large enough that $r(x_m), r(x_n) \leq (1+\delta_\varepsilon)\rho(A) \leq 2\rho(A)$. Suppose that $|x_m x_n| \geq 2\varepsilon\rho(A)$. Let y be the midpoint between x_m and x_n, then we get by uniform convexity $|ya| \leq (1-\delta_\varepsilon)\max(|x_m a|, |x_n a|) \leq (1-\delta_\varepsilon^2)\rho(A)$ for all $a \in A$. Thus $r(y) < \rho(A)$, a contradiction. So we have $|x_m x_n| < 2\varepsilon\rho(A)$, showing that x_n is a Cauchy sequence. Then the limit x satisfies $r(x) = \rho(A)$, and the same inequality shows that x is unique with this property. $\square$

Lemma 5.2.3. *Let $A \subseteq X$ be a bounded subset of a uniformly convex metric space.*

(i) Let $\varepsilon > 0$ and consider the annulus $D = \{a \in A \mid |a, Z(A)| \geq \rho(A) - \varepsilon\}$. We have $Z(D) = Z(A)$.

(ii) Let $B \subseteq A$ non-empty and $\varepsilon > 0$. Then

$$|Z(A)Z(B)| \leq \varepsilon\rho(A) + \kappa_\varepsilon(\rho(A) - \rho(B))$$

where $\kappa_\varepsilon = \frac{2-\varepsilon}{1-\exp\left(\frac{-\log(\varepsilon/2)\log(1-\delta_\varepsilon)}{\log 2}\right)}$ depends only on ε and δ_ε.

(iii) If X is a Hilbert space then $Z(A) \in \overline{\mathrm{co}}(A)$.

Proof. (i) Since $D \subseteq A$ we have $\rho(D) \leq \rho(A)$, and we also have $\sup_{d \in D} |dZ(A)| = \rho(A)$. Assume for a contradiction that $Z(D) \neq Z(A)$. Let x be a point on the geodesic $[Z(D), Z(A)]$ with $0 < |xZ(A)| < \varepsilon$. Let m be the midpoint between x and $Z(A)$. Let $\varepsilon' = \frac{|xZ(A)|}{\rho(A)}$. For $a \in D$ we have $|aZ(D)| \leq \rho(D) \leq \rho(A)$ and $|aZ(A)| \leq \rho(A)$. By convexity, $|ax| \leq \rho(A)$. By uniform convexity, $|am| \leq (1-\delta_{\varepsilon'})\rho(A)$. For $a \in A \setminus D$ we have $|aZ(A)| \leq \rho(A) - \varepsilon$ so $|am| \leq \rho(A) - \frac{\varepsilon}{2}$. This shows that m is a better centre for A then $Z(A)$, giving a contradiction.

(ii) Let $\varepsilon > 0$ and suppose $|Z(A)Z(B)| \geq \varepsilon\rho(A)\left(\frac{\rho(A)}{\rho(B)}\right)^{\frac{\log(2)}{-\log(1-\delta_\varepsilon)}}$. Let $n = \lfloor \frac{\log(\rho(A)) - \log(\rho(B))}{-\log(1-\delta_\varepsilon)} + 1 \rfloor$. Then $(1-\delta_\varepsilon)^n < \frac{\rho(A)}{\rho(B)}$, and $|Z(A)Z(B)| \geq 2^{n-1}\varepsilon\rho(A)$. Let $m_0 = Z(A)$ and for $1 \leq k \leq n$, let m_k be the midpoint between m_{k-1} and $Z(B)$. For $b \in B$ we have $|Z(B)b| \leq \rho(B)$ and $|Z(A)b| \leq \rho(A)$. By uniform convexity and induction we get $|m_k b| \leq \max((1-\delta_\varepsilon)^k\rho(A), (1-\delta_\varepsilon)\rho(B))$. We get $\sup_{b\in B} |m_n b| \leq \max((1-\delta_\varepsilon)^n\rho(A), (1-\delta_\varepsilon)\rho(B)) < \rho(B)$. Thus m_n is a better centre for B then $Z(B)$, a contradiction.

So $|Z(A)Z(B)| < \varepsilon\rho(A)\left(\frac{\rho(A)}{\rho(B)}\right)^{\frac{\log(2)}{-\log(1-\delta_\varepsilon)}}$. Since any point in B is at distance at most $\rho(A)$ from both $Z(A)$ and $Z(B)$ we also have $|Z(A)Z(B)| \leq 2\rho(A)$. Using the inequality

$\max\left(\left(\frac{1}{1-x}\right)^{\frac{\log(2)}{-\log(1-\delta_\varepsilon)}}, 2\varepsilon^{-1}\right) \leq 1+\varepsilon^{-1}\kappa_\varepsilon x$ for $x \in [0,1)$ we get $|Z(A)Z(B)| \leq \varepsilon\rho(A) + \kappa_\varepsilon(\rho(A)-\rho(B))$.

(iii) Suppose $Z(A) \notin \overline{\mathrm{co}}(A)$. Then there is a unit vector w with $\langle Z(A), w\rangle < \inf_{a\in A}\langle a, w\rangle$. But then $Z(A)+\varepsilon w$ is a better centre for A for small ε, contradiction.

□

Remark 5.2.4. If X is a Hilbert space and $B \subseteq A$ are non-empty bounded subsets we can find the simpler estimate $|Z(A)Z(B)| \leq \sqrt{\rho(A)^2 - \rho(B)^2}$.

We record the following easy fact.

Proposition 5.2.5. *Let G be a bornological group and $H \subseteq G$ a coarsely dense subgroup. Let G act isometrically and controlled on a uniformly convex metric space X and suppose there is a point in X fixed by H. Then there is also a point fixed by G.*

Proof. There is a bounded $C \subseteq G$ with $CH = G$. Let $\alpha\colon G \curvearrowright X$ be the isometric action and $x \in X$ the point which is fixed by H. Now $\alpha(G)x = \alpha(C)x$ which is bounded, so α has bounded orbits. By Lemma 5.2.2, there is a unique centre $Z(\alpha(G)x)$. This is a fixed point for G: of any $g \in G$ we have $gZ(\alpha(G)x) = Z(g\alpha(G)x) = Z(\alpha(G)x)$. □

5.2.2 Mean centre

The next variant of the centre allows us to talk about some kind of average of a mean on a uniformly convex Banach space, without additional regularity properties. It generalises both the centre of a bounded set, and the barycentre of a probability measure. This notion will be used to provide an alternative proof of Proposition 5.1.5 in the special case of an isometric action of a locally compact group on a uniformly convex Banach space. However we believe it could be useful for future applications.

Construction 5.2.6. *Let X be a closed convex subset of a uniformly convex Banach space. Let $C \subseteq X$ be a bounded subset. Let Σ be an algebra on C and let $\mu\colon \Sigma \to [0,\infty)$ be a finitely additive measure with $\mu(C) > 0$. We construct the* mean centre *$\tilde{Z}(\mu) \in X$ with the following properties:*

(i) for $A \in \Sigma$ we have

$$\tilde{Z}(\mu) = \frac{\mu(A)}{\mu(C)}\tilde{Z}(\mu_{|A}) + \frac{\mu(C\setminus A)}{\mu(C)}\tilde{Z}(\mu_{|C\setminus A});$$

(ii) For any isometry $g \in \mathrm{Isom}(X)$ we have $\tilde{Z}(g_*\mu) = g\cdot\tilde{Z}(\mu)$.

Proof. For a bounded subset $B \subseteq X$, recall the definitions of $Z(B)$ and $\rho(B)$ from Lemma 5.2.2. For any finite partition $P = (A_i)$ of C into subsets in Σ, we define

$$\tilde{Z}_P(\mu) = \frac{1}{\mu(C)}\sum_i \mu(A_i)Z(A_i)$$

and

$$\overline{\rho}_P(\mu) = \frac{1}{\mu(C)}\sum_i \mu(A_i)\rho(A_i).$$

We will define $\tilde{Z}(\mu)$ as a limit of $\tilde{Z}_P(\mu)$, so we have to show convergence as P gets more refined.

Let $\varepsilon > 0$ and κ_ε as in Lemma 5.2.3(ii). Then $\|Z(A) - Z(B)\| \leq \varepsilon\rho(A) + \kappa_\varepsilon(\rho(A) - \rho(B))$ for all $\emptyset \neq B \subseteq A \subseteq C$.

Let $P = (A_i)$ be a finite partition of C and let $Q = (B_{ij})$ be a refinement of P. Then we have

$$\begin{aligned}\left\|\tilde{Z}_P(\mu) - \tilde{Z}_Q(\mu)\right\| &= \frac{1}{\mu(C)}\left\|\sum_i \left(\sum_j \mu(B_{ij})Z(B_{ij})\right) - \mu(A_i)Z(A_i)\right\| \\ &\leq \frac{1}{\mu(C)}\sum_{i,j}\mu(B_{ij})\|Z(B_{ij}) - Z(A_i)\| \\ &\leq \frac{\varepsilon}{\mu(C)}\sum_{i,j}\mu(B_{ij})(\rho(A_i) + \varepsilon^{-1}\kappa_\varepsilon(\rho(A_i) - \rho(B_{ij}))) \\ &= \varepsilon\overline{\rho}_P(\mu) + \kappa_\varepsilon(\overline{\rho}_P(\mu) - \overline{\rho}_Q(\mu)).\end{aligned}$$

Besides, we observe that $\overline{\rho}_Q(\mu) \leq \overline{\rho}_P(\mu)$. Now let $P_1, P_2, \ldots$ be a sequence of partitions, such that P_{n+1} is a refinement of P_n, and $\overline{\rho}_{P_n}(\mu)$ converges to the infimum of $\overline{\rho}_P(\mu)$ taken over all partitions P. Then the computation above shows that $\tilde{Z}_{P_n}(\mu)$ is a Cauchy sequence: for large m, n, we will have $\overline{\rho}_{P_n}(\mu) - \overline{\rho}_{Q_n}(\mu) < \frac{\varepsilon}{\kappa_\varepsilon}$, and then $\left\|\tilde{Z}_{P_n}(\mu) - \tilde{Z}_{P_m}(\mu)\right\| < \varepsilon\rho(C) + \varepsilon$, and this works for any $\varepsilon > 0$. So the sequence $\tilde{\rho}_{P_n}(\mu)$ converges to a point $\tilde{Z}(\mu) \in X$ which is independent of the chosen sequence P_n.

Now let $A \in \Sigma$. We can choose the partitions P_n to be refinements of the partition $\{A, U \setminus A\}$. Then they split into a partition of A and one of $U \setminus A$, and we see that

$$\tilde{Z}(\mu) = \frac{\mu(A)}{\mu(U)}\tilde{Z}(\mu_{|A}) + \frac{\mu(U \setminus A)}{\mu(U)}\tilde{Z}(\mu_{|U \setminus A}).$$

The equivariance of $\tilde{Z}$ by isometries follows directly from the definition. □

Remark 5.2.7. If $\Sigma = \{\emptyset, C\}$, then $\tilde{Z}(\mu) = Z(C)$. Moreover, one can check that if C is weakly compact, and μ is a Radon probability measure on C (equipped with the weak topology), then $\tilde{Z}(\mu)$ coincides with the barycentre of μ.

Remark 5.2.8. The above construction does not quite work for any uniformly convex metric space X because there is no linear structure on X. However, instead of constructing $\tilde{Z}(\mu)$ as a point in X, it is possible by a similar construction to construct a Radon probability measure on X.

5.2.3 Shopping centres

In this section, we prove a technical result which will be central to the proof of Proposition 5.1.12, hence of Theorem 5.F. Recall that given a controlled action σ of a bornological group G with bounded geometry on a Hilbert space $\mathcal{H}$, we need to find $v \in \mathcal{H}$ such that $\sigma(A) \cdot v$ is relatively compact for all bounded $A \subset G$. Example 5.2.12 below gives a hint that simply taking the centre $Z(\sigma(A))$ for some gordo subset A does not seem to be a good approach. Instead, we will need another concept of centre which tries to ignore finite-dimensional subspaces of $\mathcal{H}$.

For a finite-dimensional subspace $V \subseteq \mathcal{H}$, let $p_{V^\perp}$ be the projection onto $V^\perp$. For a bounded set $\Lambda \subseteq \mathcal{H}$, let $s_V(\Lambda) = \rho(p_{V^\perp}(\Lambda))$. Let $s(\Lambda) = \inf_V s_V(\Lambda)$, taking the infimum over all finite-dimensional subspaces. Note that $s(\Lambda) = 0$ if and only if Λ is relatively compact.

Definition 5.2.9. Let $\Lambda \subseteq \mathcal{H}$ be a bounded subset. A point $z \in \mathcal{H}$ is called a *shopping centre* for Λ if for all finite-dimensional $V \subseteq \mathcal{H}$ and all $\varepsilon > 0$ we have $z \in \overline{\mathrm{co}}(\{v \in \Lambda \mid \left\|p_{V^\perp}(v-z)\right\| \geq s(\Lambda) - \varepsilon\})$.

We first prove the existence of shopping centres.

Lemma 5.2.10. *Every non-empty bounded subset of a Hilbert space has a shopping centre.*

Proof. Let $\Lambda \subseteq \mathcal{H}$ be bounded. We write $s_V = s_V(\Lambda)$ and $s = s(\Lambda)$. We can assume that $s > 0$, because the statement is trivial otherwise. For any finite-dimensional $V \subseteq \mathcal{H}$ and $\varepsilon > 0$ define

$$A_{V,\varepsilon} = \{a \in \mathcal{H} \mid a \in \overline{\mathrm{co}}(\{v \in \Lambda \mid \left\|p_{V^\perp}(a-v)\right\| \geq s - \varepsilon\})\}.$$

First we show that $A_{V,\varepsilon}$ is non-empty.

Let $x = Z(p_{V^\perp}(\Lambda)) \in V^\perp$. Since $\rho(p_{V^\perp}(\Lambda)) = s_V \geq s$, we have

$$x \in \overline{\mathrm{co}}(\{p_{V^\perp}v \mid v \in \Lambda,\ \|p_{V^\perp}v - x\| \geq s - \tfrac{\varepsilon}{2}\})$$

by Lemma 5.2.3(i) and (iii). So we can find $y \in \mathrm{co}(\{p_{V^\perp}v \mid v \in \Lambda,\ \|p_{V^\perp}v - x\| \geq s - \frac{\varepsilon}{2}\})$ with $\|y - x\| < \frac{\varepsilon}{2}$. Write $y = \sum_{i=1}^k c_i p_{V^\perp} v_i$ with $c_i \geq 0$ and $\sum_{i=1}^k c_i = 1$ and $v_i \in \Lambda$ and $\|p_{V^\perp}v_i - x\| \geq s - \frac{\varepsilon}{2}$ for all i. Let $a = \sum_{i=1}^k c_i v_i$. Then

$$a \in \mathrm{co}(\{v \in \Lambda \mid \|p_{V^\perp}v - x\| \geq s - \tfrac{\varepsilon}{2}\}).$$

We have $\|p_{V^\perp}a - x\| = \|y - x\| < \frac{\varepsilon}{2}$, so

$$\{v \in \Lambda \mid \|p_{V^\perp}v - x\| \geq s - \tfrac{\varepsilon}{2}\} \subseteq \{v \in \Lambda \mid \left\|p_{V^\perp}(v-a)\right\| \geq s - \varepsilon\},$$

so $a \in A_{V,\varepsilon}$.

Note that, if $\varepsilon' \leq \varepsilon$ and $V' \supseteq V$, we have $A_{V',\varepsilon'} \subseteq A_{V,\varepsilon}$. So now we already know that any finite intersection of the $A_{V,\varepsilon}$ is non-empty.

We can bound the diameter of $p_{V^\perp}A_{V,\varepsilon}$. Assume $\varepsilon < s$. Let $x = Z(p_{V^\perp}\Lambda) \in V^\perp$ and let $a \in A_{V,\varepsilon}$. Since $a \in \overline{\mathrm{co}}(\{v \in \Lambda \mid \left\|p_{V^\perp}(a-v)\right\| \geq s - \varepsilon\})$, we can find, for every $\eta > 0$, a vector $v \in \Lambda$ with $\left\|p_{V^\perp}(a-v)\right\| \geq s - \varepsilon$ and $\langle p_{V^\perp}(a-v), p_{V^\perp}(a) - x\rangle \leq \eta$. Then $\left\|x - p_{V^\perp}(v)\right\|^2 = \left\|p_{V^\perp}(a-v)\right\|^2 + \left\|x - p_{V^\perp}(a)\right\|^2 - 2\langle p_{V^\perp}(a-v), p_{V^\perp}(a) - x\rangle \geq (s-\varepsilon)^2 + \left\|x - p_{V^\perp}(a)\right\|^2 - 2\eta$. On the other hand we have $\left\|x - p_{V^\perp}(v)\right\|^2 \leq s_V^2$, so $\left\|p_{V^\perp}(a) - x\right\|^2 \leq s_V^2 - (s-\varepsilon)^2 + 2\eta$. This holds for all $\eta > 0$ so $\left\|p_{V^\perp}(a) - x\right\| \leq \sqrt{s_V^2 - (s-\varepsilon)^2}$. So the diameter of $p_{V^\perp}A_{V,\varepsilon}$ is at most $2\sqrt{s_V^2 - (s-\varepsilon)^2}$. This then also holds for the diameter of the weak closure of $p_{V^\perp}A_{V,\varepsilon}$.

Let b be in the weak closure of $A_{V,\varepsilon}$. Then for any $a \in A_{V,\varepsilon}$ we have $\left\|p_{V^\perp}(a-b)\right\| \leq 2\sqrt{s_V^2 - (s-\varepsilon)^2}$, so

$$\{v \in \Lambda \mid \left\|p_{V^\perp}(a-v)\right\| \geq s - \varepsilon\} \subseteq \{v \in \Lambda \mid \left\|p_{V^\perp}(b-v)\right\| \geq s - \varepsilon - 2\sqrt{s_V^2 - (s-\varepsilon)^2}\}.$$

This shows that

$$A_{V,\varepsilon} \subseteq \overline{\mathrm{co}}(\{v \in \Lambda \mid \left\|p_{V^\perp}(b-v)\right\| \geq s - \varepsilon - 2\sqrt{s_V^2 - (s-\varepsilon)^2}\}).$$

Since convex closed sets are also weakly closed, we find that $b \in A_{V,\varepsilon+2\sqrt{s_V^2-(s-\varepsilon)^2}}$. So the weak closure of $A_{V,\varepsilon}$ is contained in $A_{V,\varepsilon+2\sqrt{s_V^2-(s-\varepsilon)^2}}$.

This shows that any $A_{V,\varepsilon}$ also contains the weak closure of another $A_{V',\varepsilon'}$: just take ε' small enough that $\varepsilon' + 2\sqrt{(s+\varepsilon')^2 - (s-\varepsilon')^2} \leq \varepsilon$ and V' large enough that $s_{V'} \leq s + \varepsilon'$. Now the weak closure of $A_{V,\varepsilon}$ is weakly compact, and any finite intersection of them is non-empty, so there is z that is contained in the weak closure of every $A_{V,\varepsilon}$. By the above, $z \in A_{V,\varepsilon}$ for all finite-dimensional $V \subseteq \mathcal{H}$ and $\varepsilon > 0$, so z is a shopping centre for Λ. $\square$

Shopping centres are not unique, but the set of shopping centres is relatively compact. In fact we can prove a stronger result, which we will use later to prove Proposition 5.1.12.

Lemma 5.2.11. *Let $\Lambda \subseteq \mathcal{H}$ be a non-empty bounded subset. Let $f_1, \ldots, f_k \colon \mathcal{H} \to \mathcal{H}$ be isometries, and let $\Delta = \bigcup_{i=1}^k f_i(\Lambda)$. We say $z \in \mathcal{H}$ is a* possible shopping centre *if there exists an isometry $f \colon \mathcal{H} \to \mathcal{H}$ such that $f(\Lambda) \subseteq \Delta$ and z is a shopping centre of $f(\Lambda)$. Let M be the set of possible shopping centres. Then M is relatively compact.*

Proof. We write $s_V = s_V(\Lambda)$ and $s = s(\Lambda)$. Let $\varepsilon > 0$. Choose a finite-dimensional affine subspace $V \subseteq \mathcal{H}$ with $s_V \leq s + \varepsilon$ and $Z(p_{V^\perp}(\Lambda)) \in V$. Then for all $v \in \Lambda$ we have $d(v, V) \leq s + \varepsilon$.

Let W be the affine hull of the union $\bigcup_{i=1}^k f_i(V)$. This is again a finite-dimensional affine subspace. For all $v \in f_i(\Lambda)$, we have $d(v, f_i(V)) \leq s + \varepsilon$, so for all $v \in \Delta$ we have $d(v, W) \leq s + \varepsilon$.

Now let $z \in M$ be a possible shopping centre. We will show that z is close to W. Let w be the projection of z on W. To simplify notation, let us assume that $z = 0$. There is an isometry $f \colon \mathcal{H} \to \mathcal{H}$ such that $f(\Lambda) \subseteq \Delta$ and 0 is a shopping centre of $f(\Lambda)$. Then there is $v \in f(\Lambda)$ such that $\langle v, w \rangle \leq \varepsilon$ and $\left\| p_{W^\perp}(v) \right\| \geq s - \varepsilon$. Then

$$\begin{aligned}
d(v, W)^2 &= \left\| p_{W^\perp}(v - w) \right\|^2 \\
&= \left\| p_{W^\perp}(v) \right\|^2 + \left\| p_{W^\perp}(w) \right\|^2 - 2\langle p_{W^\perp}(v), w \rangle \\
&= \left\| p_{W^\perp}(v) \right\|^2 + \|w\|^2 - 2\langle v, w \rangle \\
&\geq (s - \varepsilon)\max(0, s - \varepsilon) + \|w\|^2 - 2\varepsilon.
\end{aligned}$$

On the other hand, we have $v \in \Delta$, so $d(v, W) \leq s + \varepsilon$. So we get $d(0, W) = \|w\| \leq \eta = \sqrt{4\varepsilon \max(s, \varepsilon) + 2\varepsilon}$.

So M is contained in an η-neighbourhood of W. Since M is also bounded, we can cover M with finitely many balls of radius 2η. As ε gets smaller, η gets arbitrarily small, so M is totally bounded, which is the same as relatively compact. $\square$

To justify our relatively complicated notion of shopping centre, we give an example showing that Lemma 5.2.11 is not true for the usual centre.

Example 5.2.12. Let $\mathcal{H}$ be a separable Hilbert space with basis $v_1, v_2, v_3, \ldots$. Let $B(0,1)$ denote the closed ball of radius 1 around the origin and let Λ be the bounded set $B(0,1) \cup \{10v_1 + v_2, -10v_1 + v_2\}$. Let $f_1 \colon \mathcal{H} \to \mathcal{H}$ be the identity, let f_2 be the isometry given by adding $10v_1$ and let f_3 be the isometry given by subtracting $10v_1$. Then $\Delta = B(0,1) \cup B(10v_1, 1) \cup B(-10v_1, 1) \cup \{20v_1 + v_2, -20v_1 + v_2\}$. Note that $Z(\Lambda) = v_2$. Now for $n \geq 2$ let $g_n \colon \mathcal{H} \to \mathcal{H}$ be the isometry switching v_2 and v_n while fixing all other basis vectors. Then $g_n(\Lambda) \subseteq \Delta$. The centre of $g_n(\Lambda)$ is v_n. The set $\{v_2, v_3, \ldots\}$ is not relatively compact, showing that Lemma 5.2.11 does not hold for the usual centre. Note that the origin is the unique shopping centre of Λ.

5.3 Topological versus controlled actions

In this section we will mainly be concerned by topological groups (G,τ), i.e. groups equipped with a topology making the multiplication and inverse maps continuous. We will also always assume that the topology is Hausdorff. Each topological group (G,τ) has two natural bornologies associated to it, $\mathcal{K}$ and $\mathcal{OB}$, as described in [46].

Definition 5.3.1. Let (G,τ) be a topological group. The bornology $\mathcal{K}$ consists of all relatively compact subsets of G.

This is a bornology because the product of compact sets is again compact.

Another bornology comes from considering continuous left-invariant pseudometrics on G. The following proposition is [46, Proposition 2.7]:

Proposition 5.3.2. *Let (G,τ) be a topological group and $A \subseteq G$. The following are equivalent:*

(i) For every continuous left-invariant pseudometric d on G we have $\operatorname{diam}_d(A) < \infty$.

(ii) For every continuous isometric action $\alpha\colon G \curvearrowright X$ on a metric space X, and every $x \in X$, we have $\operatorname{diam}(\alpha(A)x) < \infty$.

(iii) For every sequence of open sets $V_1 \subseteq V_2 \subseteq V_3 \cdots \subseteq G$ of open subsets with $V_n^2 \subseteq V_{n+1}$ and $\bigcup_n V_n = G$, we have $A \subseteq V_n$ for some n.

□

Definition 5.3.3. The bornology $\mathcal{OB}$ consists of all subsets A of G that satisfy the conditions of Proposition 5.3.2.

We have $\mathcal{K} \subseteq \mathcal{OB}$. For σ-compact locally compact groups, we have $\mathcal{K} = \mathcal{OB}$ (see [46, Corollary 2.8]).

We want to study the relationship between topological fixed point properties and coarse fixed point properties. We have already defined in the introduction the notions of topological versus coarse fixed point properties. Since the bornologies we shall consider will always be contained in $\mathcal{OB}$, all continuous isometric actions are controlled, so that topological FX is always stronger than coarse FX (for any metric space X).

Now if we turn to property (T), it is less clear how topological (T) should be defined: indeed, even if we focus on continuous representations, there are more than one way of defining almost fixed points. A solution is then to define topological property (T) for a topological group *equipped with a bornology* (contained in $\mathcal{OB}$).

Definition 5.3.4. A *topological bornological group* is a triple $(G,\tau,\mathcal{B})$ such that (G,τ) is a topological group and $(G,\mathcal{B})$ is a bornological group, and $\mathcal{B}$ is contained in $\mathcal{OB}(\tau)$. Let $\mathcal{E}$ be a family of Banach space. We say $(G,\tau,\mathcal{B})$ has *topological property (T$\mathcal{E}$)* if every continuous (wrt τ) isometric representation on an element of $\mathcal{E}$ with almost invariant vectors (wrt $\mathcal{B}$) has a fixed point.

For any topological group (G,τ) there are the natural bornologies $\mathcal{K}$ and $\mathcal{OB}$, and we can consider topological property (T) for the triples $(G,\tau,\mathcal{K})$ and $(G,\tau,\mathcal{OB})$ separately. For a σ-compact locally compact group (G,τ) we have $\mathcal{K} = \mathcal{OB}$, and the properties above reduce to their usual versions when we take this bornology.

5.3.1 Proof of Proposition 5.1.5

To be able to prove anything about topological bornological groups, we need some relation between the topology and the bornology. It turns out that all we need is the following.

Definition 5.3.5. A topological bornological group $(G, \tau, \mathcal{B})$ is *locally bounded* if there is a non-empty open bounded set.

For an action of G on a topological space X, let $X_{\text{cont}} = \{x \in X \mid g \to gx \text{ is continuous}\}$ be the continuous part of the action.

Lemma 5.3.6. *Let (G, τ) be a topological group acting isometrically on a uniformly convex metric space X. Then X_{cont} is an invariant closed convex subspace of X. If X is a Banach space and the action is linear, X_{cont} is also a linear subspace.*

Proof. Since the action is isometric, the continuous part is invariant and $x \in X_{\text{cont}}$ if and only if $g \to gx$ is continuous at the identity.

Let $x \in \overline{X}_{\text{cont}}$ and $\varepsilon > 0$. There is $y \in X_{\text{cont}}$ with $|xy| < \varepsilon$ and there is an open neighbourhood of the identity $U \subseteq G$ such that $|gy, y| < \varepsilon$ for all $g \in U$. Then for $g \in U$ we have $|gx, x| \leq |gx, gy| + |gy, y| + |yx| < 3\varepsilon$, showing that $x \in X_{\text{cont}}$.

To show that X_{cont} is convex, it is enough to show that if $x, y \in X_{\text{cont}}$ then the midpoint m_{xy} is in X_{cont}. Let $\varepsilon > 0$ and let U be a neighbourhood of the identity such that $g \in U$ implies $|gx, x| < \frac{1}{2}|xy|\delta_\varepsilon$ and $|gy, y| < \frac{1}{2}|xy|\delta_\varepsilon$. Let $g \in U$. Since the action is isometric we have $m_{gx,gy} = gm_{xy}$. We have $|x, gm_{xy}| \leq |x, gx| + |gx, gm_{xy}| \leq \frac{1}{2}|xy|(1 + \delta_\varepsilon)$. Let n be the midpoint between m_{xy} and gm_{xy} and suppose that $|m_{xy}, gm_{xy}| \geq \frac{1}{2}\varepsilon|xy|(1 + \delta_\varepsilon)$. By uniform convexity we have $|xn| \leq (1 - \delta_\varepsilon)\frac{1}{2}|xy|(1 + \delta_\varepsilon)$ and similarly $|yn| \leq (1 - \delta_\varepsilon)\frac{1}{2}|xy|(1 + \delta_\varepsilon)$. So $|xy| \leq |xn| + |yn| \leq |xy|(1 - \delta_\varepsilon)(1 + \delta_\varepsilon)$, a contradiction. So we have $|m_{xy}, gm_{xy}| < \frac{1}{2}\varepsilon|xy|(1 + \delta_\varepsilon)$, showing that $m_{xy} \in X_{\text{cont}}$.

Lastly, if the action is linear it is clear that X_{cont} is closed under addition and scalar multiplication. □

So the continuous part always forms a uniformly convex subaction. Of course, it might be empty, or in the linear case, consist of just the origin. However in some cases we can prove that the continuous part is non-trivial. To do this we need the following lemma. For open neighbourhoods of the identity $U, V \subseteq G$ we write $U < V$ if there is an open W such that $WU \subseteq V$. Every topological group has a large chain of open neighbourhoods.

Lemma 5.3.7. *Let G be a topological group and $U \subseteq G$ an open neighbourhood of the identity. Then there are open neighbourhoods of the identity V_t for $t \in (0, 1]$ with $V_1 = U$ and $V_t < V_{t'}$ if $t < t'$.*

Proof. Let $U_0 = U$ and for $n \geq 1$ we pick open neighbourhoods of the identity U_n recursively with $U_n^2 \subseteq U_{n-1}$. For $t \in [0, 1]$ let σ_t be the set of finite sequences of integers $\{(n_1, n_2, \ldots, n_s) \mid n_1 > n_2 > \cdots > n_s \geq 0 \text{ and } \sum_{i=1}^{s} 2^{-n_i} \leq t\}$. Define

$$V_t = \bigcup_{(n_1, n_2, \ldots, n_s) \in \sigma_t} U_{n_1} U_{n_2} \cdots U_{n_s}.$$

Note that when $t = 2^{-n_1} + 2^{-n_2} + \cdots + 2^{-n_s}$ is a dyadic rational with $n_1 > n_2 > \cdots > n_s$ we have $V_t = U_{n_1} U_{n_2} \cdots U_{n_s}$. In particular, $V_1 = U$.

For $t \leq t'$ we have $V_t \subseteq V_{t'}$. Moreover, there is a dyadic rational $2^{-n_1} + \cdots + 2^{-n_s}$ with $n_1 > \cdots > n_s$ and $t < 2^{-n_1} + \cdots + 2^{-n_s} < t'$. There is also $n_0 > n_1$ with $2^{-n_0} + 2^{-n_1} + \cdots + 2^{-n_s} < t'$. Then $U_{n_0} V_t \subseteq U_{n_0} U_{n_1} \cdots U_{n_s} \subseteq V_{t'}$. So $V_t < V_{t'}$. $\square$

Now we can show Proposition 5.1.5, which for convenience, we restate below.

Proposition 5.3.8. *Let G be a topological group and let U be an open neighbourhood of the identity. Let X be a uniformly convex metric space. Let $\alpha \colon G \curvearrowright X$ be an isometric action. Let $c > 0$ and suppose there is $x \in X$ such that $d(\alpha(g)x, x) \leq c$ for all $g \in U$. Then there is $y \in X$ with $d(x, y) \leq 2c$, such that the restriction of α to the closed convex hull of $\alpha(G)y$ is continuous.*

Proof. Let (V_t) be as in Lemma 5.3.7. We will consider the centres of the sets $\alpha(V_t)(x)$. For $t \in (0, 1)$ define $r(t) = \rho(\alpha(V_t)(x))$. This is an increasing function, so there is some $t_0 \in (0, 1)$ where r is continuous. Define $y = Z(\alpha(V_{t_0})(x))$. Since $|ax| \leq c$ for all $a \in \alpha(V_{t_0})(x)$, we have $r(\alpha(V_{t_0})(x)) \leq c$ and $|xy| \leq 2c$. We will show that the map $g \to \alpha(g)y$ is continuous. Since $|\alpha(g)y, \alpha(h)y| = |\alpha(g^{-1}h)y, y|$, it is enough to show that $|\alpha(g)y, y|$ is small for g close to the identity.

Let $\varepsilon > 0$ and $t > t_0$. Let κ_ε be as in Lemma 5.2.3(ii). There is an open W with $WV_{t_0} \subseteq V_t$. Let $g \in W$. We get

$$\alpha(g)y = Z(\alpha(g)\alpha(V_{t_0})(x)) = Z(\alpha(gV_{t_0})(x)).$$

We have $gV_{t_0} \subseteq V_t$. Since $\rho(\alpha(g)\alpha(V_{t_0})(x)) = \rho(\alpha(V_{t_0})(x)) = r(t_0)$, we get

$$|\alpha(g)y, Z(\alpha(V_t)(x))| \leq \varepsilon r(t) + \kappa_\varepsilon(r(t) - r(t_0))$$

and also

$$|y, Z(\alpha(V_t)(x))| \leq \varepsilon r(t) + \kappa_\varepsilon(r(t) - r(t_0))$$

so

$$|\alpha(g)y - y| \leq 2\varepsilon r(t) + 2\kappa_\varepsilon(r(t) - r(t_0)).$$

Since r is continuous at t_0, the right-hand side goes to zero as $\varepsilon \to 0$ and $t \to t_0$. This shows that $g \to \alpha(g)y$ is continuous, so $y \in X_{\mathrm{cont}}$. By Lemma 5.3.6, the restriction of α to the closed convex hull of $\alpha(G)y$ is continuous. $\square$

Below is an alternative proof for locally compact groups acting on uniformly convex Banach spaces. The existence of a Haar measure allows us to use a regularization procedure based on our notion of mean centre.

Proof. Assume that G is locally compact and that X is a uniformly convex Banach space. Let $K \subseteq U$ be a compact neighbourhood of the identity. Let μ be the left-invariant Haar measure. Let $f \colon U \to X$ be the orbit map $f(a) = \alpha(a)x$. We will use it to push-forward the Haar measure. Let $y = \tilde{Z}((f_{|K})_* \mu)$ be the mean centre in X (see Construction 5.2.6). Then $\|x - y\| \leq 2c$. We will show that $\left\|\alpha(g)y - y\right\|$ is small for g close to the identity.

Let $\varepsilon > 0$. Since μ is outer regular, there is $K \subseteq U' \subseteq U$ with $\mu(U') \leq (1 + \varepsilon)\mu(K)$. Let V be an open neighbourhood of the identity with $VK \subseteq U'$. Let $g \in V$. Let $\lambda_g \colon K \to gK$ be left multiplication by g. Then λ_g is a measure-preserving map and $\alpha(g) \circ f_{|K} = f_{|gK} \circ \lambda_g$ so $\alpha(g)y = \tilde{Z}((\alpha(g) \circ f_{|K})_* \mu) = \tilde{Z}((f_{|gK})_* \mu)$. We get

$$\alpha(g)y - y = \tilde{Z}((f_{|gK})_* \mu) - \tilde{Z}((f_{|K})_* \mu)$$

$$= \frac{\mu(gK \setminus K)}{\mu(K)}(\tilde{Z}((f_{|gK\setminus K})_*\mu) - \tilde{Z}((f_{|K\setminus gK})_*\mu)).$$

So
$$\|\alpha(g)y - y\| \leq \frac{4\mu(K \setminus gK)}{\mu(K)}\rho(\alpha(U)x) \leq \frac{4c(\mu(U') - \mu(K))}{\mu(K)} \leq 4c\varepsilon.$$

This shows that the map $g \to \alpha(g)y$ is continuous near the identity, so $y \in X_{\mathrm{cont}}$. By Lemma 5.3.6, the restriction of α to the closed convex hull of $\alpha(G)y$ is continuous. □

This theorem gives us some corollaries such as the ones mentioned in the introduction. To state more of them, we need first to introduce good notions of isometric actions for bornological groups.

5.3.2 Isometric actions of bornological groups

Given a bornological group G, a metric space X, and an action σ of G by isometries on X, we shall consider two natural notions of "compatibility" of the action with the bornology on G.

Definition 5.3.9. Let G be a bornological group and let σ be an action of G by isometries on a metric space X. We say that the action is controlled (resp. well controlled) if the orbits of bounded sets are bounded (resp. are relatively compact).

Note that a unitary representation is trivially controlled, but not necessarily well controlled. Given a bornological group G and a metric topological space X, a map $\varphi : G \to X$ is called proper if the preimage of a bounded set is bounded.

Definition 5.3.10. Let G be a bornological group and let σ be an action of G by isometries on a metric space X. We that σ has almost fixed points if for every $\varepsilon > 0$ and every bounded subset $B \subset G$, there exists $x \in X$ such that $d(\sigma(g)x, x) \leq \varepsilon$ for all $g \in B$.

Definition 5.3.11. Let π be a norm-preserving representation of a bornological goup G on a Banach space E. We that π has almost invariant vectors if the isometric action induced by π on the unit sphere has almost fixed points.

Definition 5.3.12. Let G be a bornological group, let $\mathcal{X}$ be a family of metric spaces, and let $\mathcal{E}$ be a family of Banach spaces.

(i) We say that G has coarse property F$\mathcal{X}$ (resp. F$\mathcal{X}^-$) if every controlled (resp. well-controlled) isometric action of G on a space from $\mathcal{X}$ has a fixed point.

(ii) We say that G has coarse property (T$\mathcal{E}$) (resp. (T$\mathcal{E}^-$)) if all its norm-preserving representations (resp. well-controlled representations) on elements of $\mathcal{E}$ with almost invariant vectors have a non-zero invariant vector.

We are now ready to list some consequences of Proposition 5.1.5, which in particular applies to locally compact groups.

Corollary 5.3.13. *Let $\mathcal{E}$ be a family of uniformly convex Banach spaces closed under taking closed linear subspaces. Let $\mathcal{X}$ be a family of uniformly convex metric spaces closed under taking convex closed subsets. Let $(G, \tau, \mathcal{B})$ be a locally bounded topological bornological group.*

(i) The topological bornological group $(G, \tau, \mathcal{B})$ has topological property $(T\mathcal{E})$ if and only if the bornological group $(G, \mathcal{B})$ has coarse property $(T\mathcal{E})$.

(ii) The topological bornological group $(G, \tau, \mathcal{B})$ has the topological fix-point property $F\mathcal{X}$ if and only if the bornological group $(G, \mathcal{B})$ has the coarse fix-point property $F\mathcal{X}$.

Proof. (i) Suppose $(G, \tau, \mathcal{B})$ has topological property $(\mathrm{T}\mathcal{E})$. Let $E \in \mathcal{E}$. Consider a representation $\pi\colon G \curvearrowright E$ with almost invariant vectors. Let U be an open bounded set in G. Let $A \in \mathcal{B}$ and $\varepsilon > 0$ and let $v \in E$ be a unit vector with $\|gv - v\| \leq \varepsilon$ for all $g \in U \cup A$. By Proposition 5.1.5, there is $w \in E_{\text{cont}}$ with $\|v - w\| \leq 2\varepsilon$. In particular $\|w\| \geq 1 - 2\varepsilon$, and for $g \in K$ we have $\|gw - w\| \leq \|gw - gv\| + \|gv - v\| + \|v - w\| \leq 5\varepsilon$, so $\left\|g\frac{w}{\|w\|} - \frac{w}{\|w\|}\right\| \leq \frac{5\varepsilon}{1-2\varepsilon}$. This shows that E_{cont} has almost invariant vectors, so it has an invariant vector, showing that $(G, \mathcal{B})$ has coarse property $(T\mathcal{E})$. The other direction is trivial.

(ii) Suppose $(G, \tau, \mathcal{B})$ has topological property $\mathrm{F}\mathcal{X}$. Let $\alpha\colon G \curvearrowright X$ be a controlled isometric action on an element of $\mathcal{X}$. Since α is controlled, we can apply Proposition 5.1.5 on any element of X to find that X_{cont} is non-empty. Then X_{cont} has a fixed point, so $(G, \mathcal{B})$ has coarse property $\mathrm{F}\mathcal{X}$. The other direction is trivial.

□

Corollary 5.3.14. *Let $\mathcal{E}$ be a family of uniformly convex Banach spaces closed under taking closed linear subspaces. Let $\mathcal{X}$ be a family of uniformly convex metric spaces closed under taking convex closed subsets. Let G be a compactly generated locally compact group, and H be a dense subgroup, and let U be a relatively compact open symmetric generating subset G.*

(i) Assume G has property $(T\mathcal{E})$. Then any norm-preserving representation of H on a space $E \in \mathcal{E}$ with $(U \cap H)$-almost invariant vectors has an invariant vector.

(ii) Assume G has property $F\mathcal{X}$. Then any affine isometric action of H on some space $X \in \mathcal{X}$ such that $(U \cap H)$-orbits are bounded has a fixed point.

Proof. We equip H with the topology τ_G induced by the topology on G, and the coarse structure $\mathcal{K}_G$ induced by the coarse structure $\mathcal{K}$ on G. This makes $(H, \tau_G, \mathcal{K}_G)$ is a locally bounded topological bornological group. Since U is open, and H is dense, for all $n \geq 1$, $(U^n \cap H) = (U \cap H)^n$. But since U is symmetric and generates G, for any compact $K \subset G$, there exists $n \in \mathbb{N}$ such that $K \subset U^n$. Combining these two remarks, we get that $K \cap H \subset (U^n \cap H)$. Hence $U^n \cap H$ generates the bornology of $(H, \mathcal{K}_G)$. Moreover, since H is dense in G, any continuous isometric action of H on some complete metric space extends to G. Hence topological $(T\mathcal{E})$ and topological $F\mathcal{X}$ for $(H, \tau_G, \mathcal{K}_G)$ are equivalent respectively to $(T\mathcal{E})$ and $F\mathcal{X}$ for G. Now we deduce from Corollary 5.3.13 that $(H, \tau_G, \mathcal{K}_G)$ has coarse $(T\mathcal{E})$ under the assumption of (i), and that $(H, \tau_G, \mathcal{K}_G)$ has coarse $F\mathcal{X}$ under the assumption of (ii). We conclude from the fact that the assumption of (i) that the representation of the bornological group $(H, \mathcal{K}_G)$ has almost invariant vectors, while the assumption of (ii) implies that the action of $(H, \mathcal{K}_G)$ is controlled. □

5.4 Applications of Proposition 5.1.5 for locally compact groups

The main goal of this section is to prove applications of Proposition 5.1.5 such as Corollary 5.1.6, as well as others in the same spirit. In order to do so, we need to consider ultraproducts of isometric actions, which will be better defined in the setting of bornological groups.

5.4.1 Ultraproduct of isometric actions of bornological groups

Let (G, S) be a group equipped with a symmetric generating subset S. A controlled isometric affine action of (G, S) on a metric space X is an action whose S-orbits are bounded. Now, let G_i be an I-indexed net of groups and for each $i \in I$, let S_i be a generating set of G_i. Let $\mathcal{U}$ be a free ultrafilter on I, and denote $(G_{\mathcal{U}}, S_{\mathcal{U}})$ the bounded ultraproduct of the net (G_i, S_i). The displacement of a controlled isometric action σ of (G, S) is the infimum over all $x \in X$ of $d(x) := \sup_{s \in S} d(\sigma(s)x, x)$. We have

$$d(x) - d(y) = \sup_{s \in S} d_X(\sigma(s)x, x) - \sup_{s \in S} d_X(\sigma(s)y, y) \leq \sup_{s \in S}(d_X(\sigma(s)x, x) - d_X(\sigma(s)y, y)).$$

By triangular inequality, we have

$$d_X(\sigma(s)x, x) \leq d_X(\sigma(s)x, \sigma(s)y) + d_X(\sigma(s)y, y) + d_X(y, x) = 2d_X(y, x) + d_X(\sigma(s)y, y).$$

We deduce that

$$d(x) - d(y) \leq 2d_X(x, y).$$

Hence $d : X \to \mathbb{R}_+$ is 2-Lipschitz, hence continuous.

Definition 5.4.1. We let $\mathcal{X}$ be a family of metric spaces. We say that $\mathcal{X}$ is stable under rescaled ultraproducts if for every I-indexed net of triples (X_i, λ_i, x_i), where $x_i \in X_i$ and $\lambda_i > 0$, every ultrafilter on I, the ultraproduct of pointed metric spaces $(X_i, \lambda_i d_{X_i}, x_i)$ belongs to $\mathcal{X}$.

Lemma 5.4.2. *Let σ be an isometric action without fixed points of (G, S) on X. Then for all $\alpha < 1$ and all $R \geq 0$, there exists x such that $\inf_{|xy| \leq Rd(x)} d(y) \geq \alpha d(x)$.*

Proof. Assume by contradiction that there exists $\alpha < 1$ and $R \geq 0$ such that for all x,

$$\inf_{|xy| \leq Rd(x)} d(y) < \alpha d(x).$$

Fix some x_0 and choose recursively $x_{n+1} \in B(x_n, Rd(x_n))$ such that $d(x_{n+1}) \leq \alpha d(x_n)$. By an immediate induction we get $d(x_n) \leq \alpha^n d(x_0)$. This implies that $|x_{n+1}x_n| \leq R\alpha^n$. Therefore (x_n) is a Cauchy sequence, and let x be its limit. We deduce by continuity of d that $d(x) = 0$, meaning that x is a fixed point: contradiction. ☐

Given some isometric actions without fixed points of groups G_i, we will construct an isometric action with positive displacement (in particular without fixed point) on an ultraproduct of these groups. Since we do not want to restrict to separable metric spaces, we will have to consider nets in full generality, rather than just sequences. For technical reasons we can only use ultrafilters that are not σ-complete. Recall that an ultrafilter $\mathcal{U}$ on a set I is not σ-complete precisely when there is a function $f \colon I \to \mathbb{N}$ such that $f^{-1}(n) \notin \mathcal{U}$ for all $n \in \mathbb{N}$. If there are no measurable cardinals, then every non-principal ultrafilter is not σ-complete, and this is consistent with ZFC. Actually, we only need the existence of suitable not σ-complete ultrafilters and we can just prove this.

Lemma 5.4.3. *Let $(I, \leq)$ be a directed set without maximum. There is a net n_i indexed on I with values on $\mathbb{N}$, and a cofinal ultrafilter $\mathcal{U}$ such that $\lim_{\mathcal{U}} n_i = \infty$.*

Proof. We will first prove there are n_i such that for all N and all i there is some $j \geq i$ with $n_j \geq N$. For each $i \in I$, we choose an element $t(i) \in I$ which is larger. This gives I the structure of a directed graph, with an arrow from i to $t(i)$. Choose a set $A \subseteq I$ that contains exactly one element of each connected component of the graph. For $i \in t^n(A)$, we put $n_i = n$. Since A contains only one element of each component, we are not trying to label an element twice, and it is well-defined. Put $n_i = 0$ for all i that did not get a number yet. Now for any $i \in I$ there is $a \in A$ in the same component. Then there is a path from a to i in the graph (possibly taking edges in the opposite direction), and this path contains a vertex that is above a and i. Then for any large enough N, we see that $t^N(a) \geq i$, and $n_{t^N(a)} = N$.

Consider the sets $\{j \mid j \geq i\} \cap \{j \mid n_j \geq N\}$ for all $i \in I, N \in \mathbb{N}$. By construction of the n_i these sets are all non-empty, and they generate a filter $\mathcal{F}$. Let $\mathcal{U}$ be any ultrafilter containing $\mathcal{F}$, then $\mathcal{U}$ is a cofinal ultrafilter converging to ∞. □

Proposition 5.4.4. *Let $\mathcal{X}$ be a family of metric spaces which is stable under rescaled ultraproducts. Let $(G_i, S_i)_{i \in I}$ be a net of groups equipped with symmetric generating subsets. For every $i \in I$, let σ_i be a controlled isometric action without fixed points of (G_i, S_i) on X_i. For each i, there exist a cofinal ultrafilter $\mathcal{U}$ on I, a space $X_{\mathcal{U}} \in \mathcal{X}$, and a controlled isometric action $\sigma_{\mathcal{U}}$ of the ultraproduct $(G_{\mathcal{U}}, S_{\mathcal{U}})$ on $X_{\mathcal{U}}$ with positive displacement.*

Proof. Let n_i and $\mathcal{U}$ be as in Lemma 5.4.3. By Lemma 5.4.2, there exist points $x_i \in X_i$ such that

$$\inf_{|x_i y| \leq n_i d_i(x_i)} d_i(y) \geq (1 - 1/n_i) d_i(x_i)$$

We can consider the ultralimit $\sigma_{\mathcal{U}} = \lim_{\mathcal{U}} \sigma_n$ which defines a controlled isometric action $\sigma_{\mathcal{U}}$ of $G_{\mathcal{U}}$ on the metric space $X_{\mathcal{U}} = \lim_{\mathcal{U}}(X_i, d_{X_i}/d_i(x_i), x_i)$. For any $y = (y_i) \in X_{\mathcal{U}}$ there is $k \in \mathbb{N}$ with $|x_i y_i| \leq k d_i(x_i)$ for all i. For all i with $n_i \geq k$ we get $d_i(y_i) \geq (1 - 1/n_i) d_i(x_i)$, so there is $s_i \in S_i$ with $|s_i y_i, y_i| \geq (1 - 2/n_i) d_i(x_i)$. Now $(s_i) \in S_{\mathcal{U}}$ and $d_{X_{\mathcal{U}}}((s_i) \cdot (y_i), (y_i)) \geq 1$. So $d(y) \geq 1$. In particular $\sigma_{\mathcal{U}}$ has positive displacement. □

We can reformulate the proposition as follows.

Corollary 5.4.5. *Let $\mathcal{X}$ be a family of metric spaces which is stable under rescaled ultraproducts. Let G_i be a net of bornological groups such that the bornology of G_i is generated by $S_i \subseteq G_i$. Suppose that the G_i do not have property ($F_{\mathcal{X}}$). Then there is a cofinal ultrafilter $\mathcal{U}$ such that the ultraproduct $G_{\mathcal{U}}$, with bornology generated by $S_{\mathcal{U}}$, does not have property ($F_{\mathcal{X}}$). Better: $G_{\mathcal{U}}$ admits a controlled isometric action with positive displacement on some element of $\mathcal{X}$.*

See also [8, Proposition 1.1] for a similar result.

5.4.2 Boundedly presented groups

Definition 5.4.6. A bornological group G is *boundedly presented* if there is a bounded symmetric set S which generated the bornology, and a set of relations R consisting of words in S with a bounded length, such that $G = \langle S \mid R \rangle$.

In this section, we show that every bornological group with FH is a quotient of a boundedly presented group with FH. More generally, we make this statement for F$\mathcal{X}$ where $\mathcal{X}$ is any family of uniformly convex metric spaces closed under taking closed convex subspaces and rescaled ultraproducts.

Theorem 5.4.7. *Let $\mathcal{X}$ be a family of uniformly convex metric spaces closed under taking closed convex subspaces and rescaled ultraproducts. Let G be a bornological group with bounded geometry and bornology generated by a single set. Assume G has property $F\mathcal{X}$. There is a boundedly presented bornological group G' with bounded geometry and property $F\mathcal{X}$ together with a quotient map $G' \twoheadrightarrow G$.*

Proof. By assumption there is a symmetric subset $S \subseteq G$ generating the bornology, which we can also take to be gordo. Then $S^2 \subseteq g_1 S \cup \cdots \cup g_k S$. Let N large enough that $g_i \in S^N$ for $1 \leq i \leq k$. Let R be the set of all words in S that evaluate to the identity in G, so that $G = \langle S \mid R \rangle$. For $n \geq N+3$, let R_n be all words in R of length at most n, and let $G_n = \langle S \mid R_n \rangle$. Note that G is a quotient of G_n. Since $n \geq N+3$, we already have $S^2 \subseteq g_1 S \cup \cdots \cup g_k S$ in G_n, showing that G_n has bounded geometry. Suppose that none of the G_n have property F$\mathcal{X}$. Then by Corollary 5.4.5, there is a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that the ultraproduct $G_{\mathcal{U}}$ does not have property F$\mathcal{X}$. We have a map $f \colon G \to G_{\mathcal{U}}$ as follows: for each $g \in G$, write $g = s_1 \cdots s_l$ with l minimal, and put $f(g) = (s_1 \cdots s_l) \in G_{\mathcal{U}}$. The choice of representation of g is not unique, but the element $s_1 \cdots s_l \in G_n$ will be unique for large enough n, thus giving a well-defined map. Now let $(g_n) \in G_{\mathcal{U}}$ be any element. Note that the g_n are all in S^l for some fixed l, and S^l is covered by finitely many left translates of S. So there are $h_1, \ldots, h_m \in G$ and for each n there is $1 \leq i_n \leq m$ and $s_n \in S$ such that the equality $g_n = h_{i_n} s_n$ holds in G. For large enough n, this inequality then holds in G_n as well. Then there is one i such that $\{n \mid i_n = i\} \in \mathcal{U}$, and $(g_n) = f(h_i) \cdot (s_n)$. So $G_{\mathcal{U}} = f(G) S_{\mathcal{U}}$, showing that G is coarsely dense in $G_{\mathcal{U}}$. By Proposition 5.2.5, we get a contradiction. So one of the G_n satisfies our criteria. $\square$

5.4.3 Marked locally compact groups

Since there exists no good notion of free group in the category of compactly generated locally compact groups, we shall rather work with a *relative* notion of marked groups in this setting.

Given a locally compact compactly generated group, a generating subset S will be called admissible if it is symmetric, compact, equal to the closure of its interior, and a neighbourhood of the identity.

Definition 5.4.8. Let G^0 be a locally compact compactly generated group and let S^0 be an admissible generating subset.

(i) A (G^0, S^0)-marked group is a pair (G, π_G), where $\pi_G : G^0 \to G$ is a continuous surjective homomorphism whose kernel does not intersect the interior of S^0.

(ii) An isomorphism $(G, \pi_G) \to (G', \pi_{G'})$ is a continuous isomorphism $\varphi : G \to G'$ such that $\varphi \circ \pi_G = \pi_{G'}$. An isomorphism class is completely determined by the kernel of π_G.

(iii) We denote by $\mathcal{M}(G^0, S^0)$ the set of isomorphism classes of (G^0, S^0)-marked groups.

(iv) We equip $\mathcal{M}(G^0, S^0)$ with the Chabauty topology on closed subgroups of G^0.

Note that the subgroups $\ker_G$ are (uniformly) discrete, in particular they are closed. Let H a closed subgroup of G. We recall a basis $(\Omega_{V,K}(H))_{K,V}$ of neighbourhoods of H for the Chabauty topology, where V are neighbourhoods of 1_{G^0} and K are compact subsets: each set $\Omega_{V,K}(H)$ consists of all closed subgroups H' such that $H \cap K \subset (H' \cap K)V$ and $H' \cap K \subset (H \cap K)V$. The space of closed subgroups of a locally compact group is well-known to be compact for the Chabauty topology. Since the condition that the subgroups avoid the interior of S^0 is a closed condition, this turns $\mathcal{M}(G^0, S^0)$ into a compact topological space.

Proposition 5.4.9. *[7, Proposition 8.A.15. L] We let G be a locally compact group, and let $S \subset G$ be an admissible generating set. Let $\hat{G}$ be the group defined by the presentation $\langle S \mid R \rangle$, where R consists of all words of the form $s_1 s_2 s_3$ such that $s_1 s_2 s_3 \equiv 1$ in G. Then $\hat{G}$ admits a unique locally compact group topology such that the projection $\pi : \hat{G} \to G$ is a homeomorphism in restriction to S.*

□

Corollary 5.4.10. *We let G be a locally compact group, and let $S \subset G$ be an admissible generating set. Then $(G, \pi_G) \in \mathcal{M}(G^0, S^0)$ for some compactly presented G^0. Moreover, every open set in $\mathcal{M}(G^0, S^0)$ containing (G, π_G) contains an element $(G', \pi_{G'})$ such that $\ker(\pi_{G'}) \subseteq \ker(\pi_G)$ and G' is compactly presented.*

Proof. The first statement is an immediate consequence of Proposition 5.4.3. For the second statement: let H_n be the normal subgroup of G^0 spanned by $\ker \pi_G \cap (S^0)^n$ and let $G_n := G^0/H_n$ for all $n \geq 2$. Clearly H_n converges to $\ker \pi_G$ in the Chabauty topology. By construction, we have $H_n = \ker \pi_{G_n}$. This shows that (G_n, π_{G_n}) converges to (G, π_G). So we can put $(G', \pi'_G) = (G_n, \pi_{G_n})$ for some large enough n. Besides, a compact presentation for $(G', \pi_{G'})$ is given by $\langle S^0 \mid R^0 \cup R_n \rangle$, where $\langle S^0 \mid R^0 \rangle$ is a compact presentation of G^0 and R_n is the set of words of length $\leq n$ in S^0 representing elements of $\ker \pi_G$. □

Lemma 5.4.11. *Let $(G_i, \pi_{G_i})_{i \in I} \in \mathcal{M}(G^0, S^0)$ be a net converging to (G, π), and let $\mathcal{U}$ be a cofinal ultrafilter on I. Denote $S_i = \pi_{G_i}(S)$, and let $(G_\mathcal{U}, S_\mathcal{U})$ be the bounded ultraproduct with respect to $\mathcal{U}$ of (G_i, S_i). There is a surjective homomorphism $p : G_\mathcal{U} \to G$ such that $p(S_\mathcal{U}) = \pi_G(S)$. Moreover* $\ker(p) \subset S_\mathcal{U}$.

Proof. Let $(G^0_\mathcal{U}, S^0_\mathcal{U})$ denote the bounded ultraproduct of the constant sequence (G^0, S^0) with respect to $\mathcal{U}$. Consider the canonical surjective homomorphism $\pi_\mathcal{U} = \lim_\mathcal{U} \pi_{G_i} : (G^0_\mathcal{U}, S^0_\mathcal{U}) \to (G_\mathcal{U}, S_\mathcal{U})$. Let N^0 be the subset of $G^0_\mathcal{U}$ consisting of sequences $g_i \in S^0$ such that $\lim_\mathcal{U} g_i = 1_{G^0}$.

We note that $N = \pi_\mathcal{U}(N^0)$ coincides with the subset of $G_\mathcal{U}$ consisting of nets $\pi_{G_i}(g_i) \in S_i$ such that $\lim_\mathcal{U} g_i = 1_G$. Since G^0 is locally compact, for all bounded nets (g_i) in G^0, there exists a unique $g \in G^0$ such that $\lim_\mathcal{U} g_i = g$, or equivalently such that $\lim_\mathcal{U} g^{-1} g_i = \lim_\mathcal{U} g_i g^{-1} = 1_{G^0}$. This implies that N^0 is a normal subgroup and that $G^0_\mathcal{U}/N^0$ is canonically isomorphic to G^0. It follows that N is a normal subgroup of $G_\mathcal{U}$. We have the following commutative diagram, which we want to complete by adding the arrow p.

$$\begin{tikzcd}
N^0 \arrow[r] \arrow[d] & G^0_\mathcal{U} \arrow[r, "p^0"] \arrow[d, "\pi_\mathcal{U}"'] & G^0 \arrow[d, "\pi_G"] \\
N \arrow[r] & G_\mathcal{U} \arrow[r, "p?"] & G
\end{tikzcd}$$

Now for all bounded nets $(k_i) \in \Pi_i G_i$, let (g_i) be a bounded net in G^0 such that $\pi_{G_i}(g_i) = k_i$, and let $k = \pi_G(g)$ where $g = \lim_\mathcal{U} g_i$. We need to show that k does not depend on the choice of lifted net (g_i), or equivalently, that if (h_i) is a bounded net such that $h_i \in \ker \pi_{G_i}$, then $\pi_G \circ p^0((h_i)) = 1_G$. By assumption, $\ker \pi_{G_i}$ converges to $\ker \pi_G$, hence $\lim_\mathcal{U} h_i \in \ker \pi_G$, so this is proved. Setting $p((g_i)) = k$, this shows that p is a well-defined morphism.

It is clear by construction that p is surjective (since p^0 is surjective). We are left to checking that $N = \ker p$. The inclusion $N \subset \ker p$ follows by commutativity of the diagram. For the other inclusion, let $(k_i) \in \ker p$. Let (g_i) be a lift of (k_i) in $G^0_{\mathcal{U}}$. We have that $p^0((g_i))$ belongs to the kernel of π_G, meaning that $\lim_{\mathcal{U}} g_i \in \ker \pi_G$. But since $\ker \pi_{G_i}$ converges to $\ker \pi_G$, this means that there exists $h_i \in \ker \pi_{G_i}$ such that $\lim_{\mathcal{U}} h_i^{-1} g_i = 1_{G^0}$. On replacing (g_i) by $(h_i^{-1} g_i)$, which is another lift of (k_i), we have that $(g_i) \in N^0$. Hence (k_i) lies in $\pi_{\mathcal{U}}(N^0) = N$. This ends the proof of the fact that $N = \ker p$.

Since $N \subset S_{\mathcal{U}}$, we also get the last statement of the lemma. □

We recall for convenience the statement of Theorem 5.B before proving it.

Theorem 5.4.12. *Let $\mathcal{X}$ be a family of uniformly convex metric spaces, closed under taking rescaled ultraproducts and closed under taking closed convex subspaces. Let G be a compactly generated locally compact group such that every continuous isometric action on an element of $\mathcal{X}$ as almost fixed points. Then G has property F$\mathcal{X}$.*

Proof. Let us argue by contradiction and assume G does not have property F$\mathcal{X}$. By Corollary 5.4.5, $G_{\mathcal{U}}$ has a controlled action σ on some $X \in \mathcal{X}$ with positive displacement. We let $p : G_{\mathcal{U}} \to G$ be the projection from Lemma 5.4.11. Since $\ker(p) \subset S_{\mathcal{U}}$ and the action is controlled, we deduce that the orbits of $\ker p$ are bounded. Hence the closed convex set Y of $\ker p$-fixed point is non-empty and $\sigma(G_{\mathcal{U}})$-invariant. Moreover the restriction of σ to Y factors through G. Applying Proposition 5.1.5, we find a closed convex subspace $Z \subset Y$ on which the action of G is continuous. Since this action is a subaction of σ it still has positive displacement: contradiction. □

Another corollary of Lemma 5.4.11 is the following.

Theorem 5.4.13. *Let $\mathcal{X}$ be a family of uniformly convex metric spaces which is stable under rescaled ultraproduct and closed under taking closed geodesic subspaces. The subset of S-marked groups (G, π_G) such that G has property F$\mathcal{X}$ forms an open subset of $\mathcal{M}(G^0, S^0)$.*

Corollary 5.4.14. *The subset of S-marked groups (G, π_G) such that G has property (T) forms an open subset of $\mathcal{M}(G^0, S^0)$.*

Proof. We shall prove that groups without F$\mathcal{X}$ form a closed subset of $\mathcal{M}(G^0, S^0)$. Let $(G_i, \pi_{G_i}) \in \mathcal{M}(G^0, S^0)$ be a net converging to (G, π_G), and let $\mathcal{U}$ be a cofinal ultrafilter as in Proposition 5.4.4. Denote $S_i = \pi_{G_i}(S)$. We deduce from Corollary 5.4.5 that the bounded ultraproduct $(G_{\mathcal{U}}, S_{\mathcal{U}})$ of the net (G_i, S_i) does not have F$\mathcal{X}$. Let σ be a controlled isometric action without fixed points of the bornological group $(G_{\mathcal{U}}, S_{\mathcal{U}})$ on some $X \in \mathcal{X}$. Since $S_{\mathcal{U}}$ has bounded orbits, the kernel N of the surjective homomorphism $p : G_{\mathcal{U}} \to G$ given by Lemma 5.4.11 has bounded orbits, hence admits a fixed point v. Since N is normal in $G_{\mathcal{U}}$, the set X_N of $\sigma(N)$-fixed points is a closed $\sigma(G_{\mathcal{U}})$-invariant subspace of X. We observe that in a uniformly convex metric space, any pair of points is joined by a unique geodesic, hence it follows that the set of $\sigma(N)$-fixed points is convex, and therefore $X_N \in \mathcal{X}$. The restriction of σ to X_N induces an action of G, and by Proposition 5.1.5, we may find a closed convex subspace of X_N on which the action of G is continuous. Hence G does not have property F$\mathcal{X}$ and we are done. □

Finally, we restate Theorem 5.C before proving it.

Theorem 5.4.15. *Let $\mathcal{X}$ be a family of uniformly convex metric spaces, closed under taking rescaled ultraproducts and closed under taking closed convex subspaces. Let G be a compactly generated locally compact group with property F$\mathcal{X}$. Then there exists a compactly presented locally compact group G' with property F$\mathcal{X}$, and a continuous surjective homomorphism $G' \to G$ with discrete kernel.*

Proof. Let $S \subset G$ be a compact symmetric generating neighbourhood of the identity. We then conclude by applying Corollary 5.4.10 and Theorem 5.4.13. $\square$

5.5 A focus on actions on Hilbert spaces

The main goal of this section is to relate our two versions of coarse property (T) and coarse property FH, and to prove Proposition 5.1.12. The first one is an adaptation the classical argument by Delorme and Guichardet, and while Proposition 5.1.12 might hold for actions on more general Banach spaces, we were only able to prove it for Hilbert spaces.

5.5.1 Delorme-Guichardet for bornological groups

Just like in the topological case, there is a connection between properties (T) and FH. The proof is similar to the proof of the Delorme-Guichardet Theorem, as seen in [5]. We say the bornology of a bornological group G is generated by a single set if there is some bounded set A such that for each bounded set $C \subseteq G$, there is some n with $C \subseteq A^n$. In particular, A generates G as a group.

Proposition 5.5.1. *Let G be a bornological group.*

(i) If G has coarse property (T-) (resp. (T)) then it has coarse property FH- (resp. FH).

(ii) If G has coarse property FH- (resp. FH) and its bornology is generated by a single set, then it has coarse property (T-) (resp. (T)).

Proof. First suppose that G has coarse property (T-). As in [5, Proposition 2.11.1], we ca construct a real Hilbert space $\mathcal{H}_t$, a map $\Phi_t\colon \mathcal{H} \to \mathcal{H}_t$ and an orthogonal representation $\pi_t\colon G \to O(\mathcal{H}_t)$ satisfying

$$\langle \Phi_t(v), \Phi_t(w)\rangle = \exp(-t\|v-w\|^2)$$

and

$$\pi_t(g)\Phi_t(v) = \Phi_t(\alpha(g)v)$$

and the linear span of $\Phi_t(\mathcal{H})$ is dense in $\mathcal{H}_t$.

Consider the direct sum of all these representations $\bigoplus_{t>0} \pi_t\colon G \to O(\bigoplus_{t>0} \mathcal{H}_t)$. This has almost invariant vectors: let $A \subseteq G$ be bounded and $\varepsilon > 0$. Consider the vector $\Phi_t(0)$ for some $t > 0$. It satisfies

$$\langle \pi_t(a)\Phi_t(0), \Phi_t(0)\rangle = \langle \Phi(t)(\alpha(a)(0)), \Phi_t(0)\rangle = \exp(-t\|\alpha(a)(0)\|^2)$$

for each $a \in A$. Since α is controlled, we can choose $t > 0$ such that the right-hand side is always at least $1-\varepsilon$. Then $\|\pi_t(a)\Phi_t(0) - \Phi_t(0)\|^2 = 2 - 2\langle \pi_t(a)\Phi_t(0), \Phi_t(0)\rangle \leq 2\varepsilon$, showing that $\Phi_t(0)$ is an almost invariant vector.

Since G has coarse property (T-) there is some invariant vector. This has to have a non-zero coordinate in some $\mathcal{H}_t$, so there is some $t > 0$ for which a non-zero invariant vector $w \in \mathcal{H}_t$ exists. Suppose the orbits of α are unbounded. Let $v \in \mathcal{H}$ be a vector and let $g_n \in G$ be a sequence such that $\{\alpha(g_n)v\}$ is unbounded. Then the sequence $(\Phi_t(\alpha(g_n)v))$ converges weakly to 0. For each n we have

$$\langle w, \Phi_t(v)\rangle = \langle w, \pi_t(g_n)(\Phi_t(v))\rangle = \langle w, \Phi_t(\alpha(g_n)v)\rangle.$$

Taking the limit as $n \to \infty$ shows $\langle w, \Phi_t(v) \rangle = 0$. Since $\Phi_t(\mathcal{H})$ is dense in $\mathcal{H}_t$ it follows that $w = 0$. This is a contradiction, so α has bounded orbits, and has a fixed point as well. So G has property FH.

Suppose G has property (T-). We use the same construction as above, but we start with an affine action α that is well controlled, and we need the representation π_t to be controlled. So let $A \subseteq G$ be bounded and $w \in \mathcal{H}_t$. We need to show that $\pi_t(A)w$ is relatively compact. First consider the case that $w = \Phi_t(v)$ for some $v \in \mathcal{H}$. The map $\Phi_t \colon \mathcal{H} \to \mathcal{H}_t$ is continuous, and $\alpha(A)v$ is relatively compact, so $\pi(A)w = \Phi_t(\alpha(A)v)$ is relatively compact as well. Now if w is in the span of $\Phi_t(\mathcal{H})$ it follows that $\pi(A)w$ is still relatively compact. Finally, since the span of $\Phi_t(\mathcal{H})$ is dense in $\mathcal{H}_t$ it follows that $\pi(A)w$ is relatively compact for all $w \in \mathcal{H}_t$. The rest of the argument is the same as above and it follows that G has coarse property FH-.

Now suppose the bornology of G is generated by some bounded set A, and that G has coarse property FH. Let $\pi \colon G \to O(\mathcal{H})$ be an orthogonal representation to a real Hilbert space $\mathcal{H}$, and suppose that it has almost invariant vectors, but no invariant vector. Then for any natural number n we can find a unit vector $v_n \in \mathcal{H}$ with $\|\pi(a)v_n - v_n\| \leq 2^{-n}$ for all $a \in A$. Now define the map $b \colon G \to \bigoplus_n \mathcal{H}$ by $b(g) = (\pi(g)v_n - v_n)$. We have $b(gh) = \pi(g)b(h) + b(g)$, and for $a \in A$, we have $\|b(a)\| < 1$ by the inequality above. If $C \subseteq G$ is bounded, there is some n with $C \subseteq A^n$, and it follows that $\|b(c)\| < n$ (in particular, the norm is finite, and b is well-defined).

Now define the affine action $\alpha \colon G \curvearrowright \bigoplus_n \mathcal{H}$ by $\alpha(g) = \bigoplus_n \pi(g) + b(g)$. We see that α is controlled, so it has a fixed point (w_n). For large enough n we must have $\|w_n\| < 1$. For such n we have $\alpha(w_n) = w_n$, so $w_n = \pi(g)(w_n) + \pi(g)(v_n) - v_n$ for all $g \in G$. This gives $\pi(g)(w_n + v_n) = w_n + v_n$. But π has no fixed point, so $w_n = -v_n$, which as norm 1, a contradiction. So G has coarse property (T).

Lastly, suppose the bornology of G is generated by some bounded set A, and that G has coarse property FH-. We use the same construction again, and we just have to show that if π is controlled, then α is well controlled. So let $(\eta_n) \in \bigoplus_n \mathcal{H}$ and let $C \subseteq G$ be bounded. For each n, the set $\{\pi(C)\eta_n\}$ is relatively compact. So the set $\{(\pi(c)\eta_n) \mid c \in C\}$ is also relatively compact. The set $\pi(C)\xi_n$ is also relatively compact for each n, so $\{(\pi(c)(\xi_n) - \xi(n) \mid c \in C\}$ is relatively compact as well. It follows that $\alpha(C)(\eta_n) = \{(\pi(c)\eta_n + \pi(c)(\xi_n) - \xi(n) \mid c \in C\}$ is relatively compact. So α is well controlled. The rest of the argument is the same as before, and it follows that G has coarse property (T-). $\square$

For bornological groups with bounded geometry, coarse property (T-) implies already that the bornology is generated by a bounded set.

Proposition 5.5.2. *Let G be a bornological group with bounded geometry. Suppose G has coarse property (T-). Then the bornology is generated by a bounded set.*

Proof. Let (A, ε) be a Kazhdan pair. We can assume that A is also gordo. Let H be the subgroup generated by A. Consider the representation $\pi \colon G \to U(l^2(G/H))$, given on basis vectors e_{gH} by $\pi(g')e_{gH} = e_{g'gH}$. We show that this is a controlled representation. Let $C \subseteq G$ be bounded and consider a basis vector $e_{gH} \in l^2(G/H)$. since A is gordo, we can find $g_1, \ldots, g_n$ with $Cg \subseteq g_1 A \cup \cdots \cup g_n A$. Then $\pi(C)e_{gH} = \{e_{cgH} \mid c \in C\} \subseteq \{e_{g_i H} \mid 1 \leq i \leq n\}$ is finite. It follows that $\pi(C)\xi$ is compact for all $\xi \in l^2(G/H)$, so π is a controlled representation.

The basis vector $e_H \in l^2(G/H)$ satisfies $\pi(a)e_H = e_H$ for all $a \in A$. Since (A, ε) is a Kazhdan pair, it follows that the representation has a constant vector. But this means that G/H is finite. So if we add finitely many elements to A, we can generate the whole group with a bounded set A'.

To finish the proof we need to use bounded geometry again. For a bounded set C we can write $C \subseteq g_1 A \cup \cdots \cup g_n A$. We can find k such that $g_i \in A^k$ for $1 \leq i \leq n$. Then $C \subseteq A^{k+1}$. So the bornology is generated by a bounded set. $\square$

5.5.2 Proof of Theorem 5.F

Let us recall the statement of Proposition 5.1.12.

Proposition 5.5.3. *Let G be a bornological group with bounded geometry and let $\alpha\colon G \curvearrowright \mathcal{H}$ be a controlled affine action on a Hilbert space. There is a non-empty invariant closed subspace $\mathcal{H}'$ such that the restriction of α to $\mathcal{H}'$ is well controlled.*

Proof. Let $C \subseteq G$ be gordo. Let $\Lambda = \alpha(C)(0)$. Let v be a shopping centre of Λ. We will show that $\alpha(D)(v)$ is relatively compact for all bounded $D \subseteq G$.

Since C is gordo we can write $DC \subseteq g_1 C \cup g_2 C \cup \cdots \cup g_k C$, with $g_i \in G$. Let $\Delta = \alpha(DC)(0) \subseteq \bigcup_{i=1}^{k} \alpha(g_i)\Lambda$. Every element of $\alpha(D)(v)$ can be written as $\alpha(g)v$ with $g \in D$, and this is a shopping centre of $\alpha(g)\Lambda \subseteq \Delta$. By Lemma 5.2.11, we see that $\alpha(D)(v)$ is relatively compact.

Now it follows easily that $\alpha(D)w$ is relatively compact for all $w \in \alpha(G)v$. Then this also holds for w in the affine hull of $\alpha(G)v$. Let $\mathcal{H}'$ be the closure of the affine hull of $\alpha(G)v$. Then it follows that the restriction of α to $\mathcal{H}'$ is well controlled. $\square$

Theorem 5.5.4. *For bornological groups with bounded geometry, coarse (T) and coarse (T-) are equivalent. Similarly, coarse FH and coarse FH- are equivalent for these groups.*

Proof. Suppose G has coarse property FH-. Consider a controlled affine action $\alpha\colon G \curvearrowright \mathcal{H}$. By Proposition 5.1.12, there is a non-empty closed invariant subspace $\mathcal{H}'$ such that the restriction of α to $\mathcal{H}'$ is well controlled. Then there is a fixed point in $\mathcal{H}'$. So G has coarse property FH.

Now note that G has coarse property (T) if and only if its bornology is generated by a bounded set and G has coarse property FH, while G has coarse property (T-) if and only if its bornology is generated by a bounded set and G has coarse property FH-. So coarse (T) and coarse (T-) are also equivalent. $\square$

We will use this theorem to prove the following result. For a bornological group G and an ultrafilter $\mathcal{U}$ on any index set I, we consider the bounded ultraproduct $G_{\mathcal{U}}$ which is the quotient of bounded sequences $\{(g_n) \mid \{g_n\} \text{ is bounded}\}$ by the equivalence relation $(g_n) \sim (h_n)$ if $\{n \mid g_n = h_n\} \in \mathcal{U}$. This has a natural bornology generated by the images of A^I for bounded $A \subseteq G$.

Proposition 5.5.5. *Let G be a bornological group with bounded geometry and let $\mathcal{U}$ be an ultrafilter on I. Then G has coarse property FH if and only if $G_{\mathcal{U}}$ has coarse property FH.*

Proof. Let $C \subseteq G$ be a gordo set. Assume that G has property coarse (FH). We have an embedding $j : G \to G_{\mathcal{U}}$. Consider $(g_i) \in G_{\mathcal{U}}$. Since all g_i must be contained in a bounded set and C is gordo, we can find finitely many $h_1, \ldots, h_n \in G$ such that for all i there is $1 \leq k_i \leq n$ and $c_i \in C$ with $g_i = h_{k_i} c_i$. Then there is one $1 \leq k \leq n$ such that $\{i \mid k_i = k\} \in \mathcal{U}$. We find $(g_i) = h_k(c_i) \in j(G)C_{\mathcal{U}}$. Thus $j(G)C_{\mathcal{U}} = G_{\mathcal{U}}$, showing that G is coarsely dense in $G_{\mathcal{U}}$. Now consider a controlled action $\alpha\colon G_{\mathcal{U}} \curvearrowright \mathcal{H}$ on a Hilbert space. Precomposing with j yields a controlled action of G, which must have a fixed point v. Then the orbit $\alpha(G_{\mathcal{U}})v$ is bounded, so its centre must be a fixed point for α. So $G_{\mathcal{U}}$ has coarse property (FH).

Now suppose that $G_{\mathcal{U}}$ has coarse property (FH). We will show that G has coarse property (FH-), which will be enough by Theorem 5.F. So let $\alpha\colon G \curvearrowright \mathcal{H}$ be a well-controlled action. We define the new representation $\beta\colon G_{\mathcal{U}} \curvearrowright \mathcal{H}$ by the formula

$$\beta((g_i))v = \lim_{\mathcal{U}} \alpha(g_i)v.$$

For any element (g_i) of $G_{\mathcal{U}}$ the set $\{(g_i)\}$ is bounded, so the set $\{\alpha(g_i)v\} \subseteq \mathcal{H}$ is relatively compact, showing that the limit on the right hand side exists. So β is well-defined, and then it is straightforward to check that it defines a well-controlled affine action. Therefore β has a fixed point v, which is then a fixed point for α as well. $\square$

5.5.3 Proof of Theorem 5.D

We need the following well-known lemma.

Lemma 5.5.6. *Let $\pi\colon G \curvearrowright \mathcal{H}$ be a unitary representation of a group G. Denote by p^G the projection on the G-invariant vectors. Then $p^G v$ belongs to the closed convex hull $\pi(G)v$ for all $v \in \mathcal{H}$.*

Proof. There exists a unique vector v_0 of minimal norm in the closed convex hull of $\pi(G)v$, which is then $\pi(G)$-invariant. For all $w \in \pi(G)v$ we have that $w - v$ is perpendicular to the G-invariant vectors. So this holds for v_0 as well, showing that $v_0 = p^G v$. $\square$

Theorem 5.5.7. *Let $(G, \mathcal{B})$ be a bornological group with bounded geometry, whose bornology is generated by a single set. Let $p\colon G \to Q$ be a surjective morphism and equip Q with the induced bornology $p(\mathcal{B})$. Assume that the kernel $A = \ker(p)$ is central in G and coarsely contained in the commutator $[G, G]$. Then G has coarse property (T) if and only if Q has coarse property (T).*

Proof. If G has property (T) then so does Q, since it is a quotient. So suppose Q has property (T), and therefore also property (FH). Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. First we show that any affine action of the bounded ultraproduct $G_{\mathcal{U}}$ on a Hilbert space $\mathcal{H}$ has almost invariant vectors. Note that we have a surjective morphism $G_{\mathcal{U}} \to Q_{\mathcal{U}}$ with central kernel $A_{\mathcal{U}}$.

So let $\sigma\colon G_{\mathcal{U}} \curvearrowright \mathcal{H}$ be an affine transformation. By Proposition 5.1.12 we may assume that σ is well controlled. We write $\sigma(g)v = \pi(g)v + b(v)$ where π is a representation and b is a cocycle for π, i.e. it satisfies $b(gh) = \pi(g)b(h) + b(g)$ for $g, h \in G_{\mathcal{U}}$.

Let $\mathcal{H}^G$ be the subspace of $\mathcal{H}$ consisting of $\pi(G_{\mathcal{U}})$-invariant vectors and let b^G denote the projection of b onto $\mathcal{H}^G$. Then the cocycle condition gives $b^G(gh) = b^G(g) + b^G(h)$, so $b^G\colon G_{\mathcal{U}} \to \mathcal{H}^G$ is a morphism to an abelian group. Since A has bounded geometry, we can prove similarly to the proof of 5.5.5 that A is coarsely dense in $A_{\mathcal{U}}$. Since A is coarsely contained in $[G, G]$ it follows that $A_{\mathcal{U}}$ is also coarsely contained in $[G_{\mathcal{U}}, G_{\mathcal{U}}]$. Since b^G is a controlled map we find that b^G is bounded on $A_{\mathcal{U}}$. But then for any $a \in A_{\mathcal{U}}$ and $n \in \mathbb{Z}$ we have $b^G(a^n) = nb^G(a)$ and boundedness implies that b^G is 0 on $A_{\mathcal{U}}$.

Now let $\mathcal{H}^A$ be the subspace of $\mathcal{H}$ consisting of $\pi(A_{\mathcal{U}})$-invariant vectors and let p^A be the projection onto $\mathcal{H}^A$ and $b^A = p^A \circ b$. The projection of the cocycle condition gives $b^A(gh) = \pi(g)b^A(h) + b^A(g)$ for $g, h \in G_{\mathcal{U}}$. Using that $A_{\mathcal{U}}$ is central in $G_{\mathcal{U}}$ we find, for $a \in A_{\mathcal{U}}$ and $g \in G_{\mathcal{U}}$:

$$b^A(g) + b^A(a) = \pi(a)b^A(g) + b^A(a) = b^A(ag) = b^A(ga) = \pi(g)b^A(a) + b^A(g)$$

so $b^A(a)$ is a $\pi(G_\mathcal{U})$-invariant vector. This implies that $b^A(a) = b^G(a) = 0$. The cocycle condition then gives $b^A(ga) = b^A(a)$ for $g \in G_\mathcal{U}$ and $a \in A_\mathcal{U}$. Hence b^A factorizes through a cocycle $b' \colon Q_\mathcal{U} \to \mathcal{H}^A$. This gives a controlled affine action of $Q_\mathcal{U}$ on $\mathcal{H}^A$. Since $Q_\mathcal{U}$ has property (FH) it follows that there is an invariant vector $v_0 \in \mathcal{H}^A$. This vector satisfies $p^A(\sigma(g)v_0)) = v_0$ for all $g \in G_\mathcal{U}$.

Let $B \subseteq G_\mathcal{U}$ be a bounded set and $\varepsilon > 0$. Since σ is well controlled, $\sigma(B)(v_0) \subseteq (p^A)^{-1}(v_0)$ is relatively compact, so we may select finitely many $v_1, \ldots, v_k \in (p^A)^{-1}(v_0)$ such that $\sigma(B)(v_0)$ is contained in a union of balls $\bigcup_{1 \leq j \leq k} B(v_j, \frac{\varepsilon}{2})$. We apply Lemma 5.5.6 to the representation $A_\mathcal{U} \curvearrowright \mathcal{H}^k$ and the vector $(v_1, \ldots, v_k)$. We find $a_1, \ldots, a_n \in A$ and $t_1, \ldots, t_n \in [0,1]$ with sum 1 such that $\left\|\sum_{i=1}^n t_i \pi(a_i) v_j - v_0\right\| < \frac{\varepsilon}{2}$ for $1 \leq j \leq k$. Let $w = \sum_{i=1}^n t_i \sigma(a_i)(v_0)$. Then for $g \in B$ we have $\left\|\sigma(g)(v_0) - v_j\right\| < \frac{\varepsilon}{2}$ for some j, and then

$$\begin{aligned}
\left\|\sigma(g)w - w\right\| &= \left\|\sum_{i=1}^n (t_i \sigma(g a_i)(v_0) - t_i \sigma(a_i)(v_0))\right\| \\
&= \left\|\sum_{i=1}^n (t_i \sigma(a_i)\sigma(g)(v_0) - t_i \sigma(a_i)(v_0))\right\| \\
&= \left\|\sum_{i=1}^n t_i \pi(a_i)(\sigma(g)(v_0) - v_0)\right\| \\
&= \left\|\sum_{i=1}^n t_i \pi(a_i)(\sigma(g)(v_0)) - v_0\right\| \\
&\leq \left\|\sum_{i=1}^n t_i \pi(a_i) v_j - v_0\right\| + \tfrac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

So w is an almost invariant vector for B. This shows that any affine controlled representation of $G_\mathcal{U}$ has almost invariant vectors.

Now to finish the proof we use Proposition 5.4.4: if G would not have property (FH), we find a controlled affine action of the ultraproduct $G_\mathcal{U}$ with positive displacement, and thus not with almost invariant vectors. So G must have property (FH) and hence property (T) since it has bounded geometry and its bornology is generated by a single set. $\square$

5.5.4 The group $\mathrm{Homeo}^+_\mathbb{Z}(\mathbb{R})$

Let $\mathrm{Homeo}^+_\mathbb{Z}(\mathbb{R})$ be the group of lifts of orientation-preserving homeomorphisms of the unit circle, as described in [46, §3.4]. It can be described explicitly as

$$\mathrm{Homeo}^+_\mathbb{Z}(\mathbb{R}) = \{ f \colon \mathbb{R} \to \mathbb{R} \text{ homeomorphism} \mid f(x+n) = f(x) + n \text{ for all } x \in \mathbb{R}, n \in \mathbb{Z}\}.$$

Rosendal shows that the bornology $\mathcal{OB}$ can be described as follows: a subset $A \subseteq \mathrm{Homeo}^+_\mathbb{Z}(\mathbb{R})$ is bounded exactly when the set $\{a(0) \mid a \in A\} \subseteq \mathbb{R}$ is bounded. In particular it follows that

$\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ is coarsely equivalent to $\mathbb{Z}$. We can use Theorem 5.D to prove that this group has property (T) as a bornological group.

Corollary 5.5.8. *The group $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ equipped with $\mathcal{OB}$ has coarse property (T). Hence, it also has coarse property FH (and in particular topological property FH).*

Proof. There is a short exact sequence of Polish groups:

$$1 \to \mathbb{Z} \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}^+(\mathbb{R}/\mathbb{Z}) \to 1,$$

where $\mathbb{Z}$ is central. Here $\mathbb{Z}$ has the discrete bornology and $\mathrm{Homeo}^+(\mathbb{R}/\mathbb{Z})$ is bounded. From the explicit description of the bornology on $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ it is clear that $\mathrm{Homeo}^+(\mathbb{R}/\mathbb{Z})$ is the image of a single bounded set. Moreover, the bornology of $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ is generated by the single bounded set $S = \{a \in \mathrm{Homeo}^*_{\mathbb{Z}}(\mathbb{R}) \mid 0 \le a(0) \le 1\}$.

For $x \in \mathbb{R}$ denote by $\{x\}$ the fractional part of x. Consider $a, b \in \mathrm{Homeo}^*_{\mathbb{Z}}(\mathbb{R})$ given by $a(x) = x - \frac{1}{4}$ and $b(x) = x + 2\{x\}$ if $\{x\} \le \frac{1}{4}$ and $b(x) = x + \frac{2}{3} - \frac{2}{3}\{x\}$ if $\{x\} \ge \frac{1}{4}$. Then $[a,b](0) = \frac{1}{2}$ and $[b^{-1}, a^{-1}](\frac{1}{2}) = 1$. So $[b^{-1}, a^{-1}][a,b]$ sends 0 to 1. Then $\big([b^{-1}, a^{-1}][a,b]\big)^n$ sends 0 to n. So the collection $\{\big([b^{-1}, a^{-1}][a,b]\big)^n \mid n \in \mathbb{Z}\}$ is coarsely dense in $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$, so in particular, the commutator subgroup is coarsely dense. So the conditions of Theorem 5.D apply, and $\mathrm{Homeo}^+(\mathbb{R}/\mathbb{Z})$ has coarse property (T). It also follows that this group has topological property FH, because this is a weaker statement. $\square$

In the same chapter of his book, Rosendal considers two other similar examples of Polish groups:

– the group $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q})$ of order-preserving permutations of $\mathbb{Q}$ commuting with integral translations, equipped with the permutation group topology.

– the subgroup $AC^*_{\mathbb{Z}}(\mathbb{R})$ of $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ consisting of all h so that h and h^{-1} are absolutely continuous when restricted to $[0,1]$, equipped with a certain topology (see [22]).

For similar reasons, these two groups equipped with $\mathcal{OB}$ are quasi-isometric to $\mathbb{Z}$, and satisfy the assumptions of Theorem 5.D, and so they have coarse property (T) as well.

5.6 Bounded products

In this section we prove Proposition 5.1.14 and Theorems 5.G and 5.H.

Proposition 5.6.1. *Let Γ_n be a sequence of groups with finite sets of generators S_n. Let $G = \prod_n \{G_n, S_n\}$ be their bounded product, with the bornology $\mathcal{B}$ and topology τ introduced in the Introduction. Then we have $\mathcal{B} = \mathcal{OB} = \mathcal{K}$.*

Proof. Let $A \in \mathcal{B}$. We show that A is relatively compact for τ by showing that every ultrafilter $\mathcal{U}$ on $\overline{A}$ has a limit point. Since A is bounded there is an integer L such that for all $(a_n) \in A$ we have $a_n \in S_n^L$ for all n. Then we can define the ultralimit $c_n = \lim_{\mathcal{U}} a_n \in S_n^L$. Then $(c_n) \in \overline{A}$. We show that it is a limit of $\mathcal{U}$. Let $(c_n) \cdot U_\rho$ be an open set containing (c_n). Then there is an N such that $\rho(n) \ge 2L$ for all $n > N$. Then $(c_n) \cdot U_\rho \cap \overline{A} \supseteq \{(a_n) \in \overline{A} \mid a_n = c_n \text{ for all } n \le N\}$, and since the latter is in $\mathcal{U}$, so is the former, showing that $\mathcal{U}$ converges to (c_n) and that $A \in \mathcal{K}$.

We always have $\mathcal{K} \subseteq \mathcal{OB}$ so it remains to show that $\mathcal{OB} \subseteq \mathcal{B}$. We use Proposition 5.3.2(iii). Suppose that $A \notin \mathcal{B}$. Then for any positive integer i we can find $((a_i)_n) \in A$ and $n_i \in \mathbb{N}$ with

$l((a_i)_{n_i}) > i2^i$. Let $\rho\colon \mathbb{N} \to \mathbb{Z}_{\geq 1}$ be a proper function satisfying $\rho(n_i) \leq i$ for all positive integers i. For $k \geq 1$ let $V_k = U_{2^k\rho}$. Then $V_k^2 \subseteq V_{k+1}$ and $\bigcup_k V_k = G$. For all i we have $((a_i)_n) \in A$, but $l((a_i)_{n_i}) > i2^i \geq 2^i \rho(n_i)$ so $((a_i)_n) \notin V_i$. So $A \not\subseteq V_i$, showing that $A \notin \mathcal{OB}$.

□

We recall the statement of Theorem 5.G:

Theorem 5.6.2. *Let Γ_n be a sequence of finite groups with symmetric generator sets S_n. Suppose that* $\sup_n |S_n| < \infty$*. Let X be the coarse disjoint union of the Cayley graphs* $\mathrm{Cay}(\Gamma_n, S_n)$*. Then the bounded product G has topological property (T) if and only if X is an expander sequence.*

Proof. Suppose G has topological property (T). For each n, let $L_0^2\Gamma_n$ denote the Hilbert space of functions on Γ_n with sum zero. The group Γ_n acts by multiplication on $L_0^2\Gamma_n$. Precomposing with the projection $G \to \Gamma_n$ gives a representation of G on $L_0^2\Gamma_n$. Taking the direct sum gives a representation of G on $\bigoplus_n L_0^2\Gamma_n$. This is a continuous representation without any invariant vectors. By property (T), there are also no almost invariant vectors. This shows that X is an expander sequence.

Conversely, suppose that X is an expander sequence. Let M_n be the normalized adjacency matrix of Γ_n. There is a constant $h > 0$ such that the spectrum $\sigma(M_n)$ is contained in $[h-1, 1-h] \cup \{1\}$ for all n. Then the normalized adjacency matrix of the product group $\Gamma_1 \times \cdots \times \Gamma_N$ is $M_1 \otimes \cdots \otimes M_N$. Its spectrum is still contained in $[h-1, 1-h] \cup \{1\}$. For any representation $\Gamma_1 \times \cdots \times \Gamma_N \curvearrowright \mathcal{H}$, denote by $p_{\leq N}$ the projection on the invariant vectors. Then we get

$$\begin{aligned} h\big\|p_{\leq N}v - v\big\| &\leq \big\|v - p_{\leq N}v\big\| - \big\|(M_1 \otimes \cdots \otimes M_N)(v - p_{\leq N}v)\big\| \\ &\leq \big\|(M_1 \otimes \cdots \otimes M_N)v - v\big\| \\ &\leq \sup_{s \in S_1 \times \cdots \times S_N} \|sv - v\|\,. \end{aligned}$$

Now let $\pi\colon G \to U(\mathcal{H})$ be a continuous representation with almost invariant vectors. Let $C = \prod_n S_n \subseteq G$. Let $k_0 = \lceil 2h^{-1} \rceil$. Since the representation has almost invariant vectors, there is a unit vector $v_{k_0} \in \mathcal{H}$ such that $\|sv_{k_0} - v_{k_0}\| \leq \frac{1}{k_0}$ for all $s \in C$. We will recursively construct unit vectors $v_k \in \mathcal{H}$ for $k \geq k_0$, with $\|sv_k - v_k\| \leq \frac{1}{k}$ for $s \in C$, and $\|v_{k+1} - v_k\| \leq \frac{4}{hk^2}$. Suppose v_k exists. Since the representation is continuous, there is an integer N such that for every $s \in \prod_{n>N} S_n$, we have $\|sv_k - v_k\| < \frac{1}{k(k+1)}$. Let

$$v_{k+1} = \frac{\left(\frac{k-2}{k} + \frac{2}{k}p_{\leq N}\right)v_k}{\left\|\left(\frac{k-2}{k} + \frac{2}{k}p_{\leq N}\right)v_k\right\|}.$$

We know that $\big\|p_{\leq N}v_k - v_k\big\| \leq \frac{1}{hk}$, so $\left\|\left(\frac{k-2}{k} + \frac{2}{k}p_{\leq N}\right)v_k - v_k\right\| \leq \frac{2}{hk^2}$. In particular we get $\left\|\left(\frac{k-2}{k} + \frac{2}{k}p_{\leq N}\right)v_k\right\| \geq 1 - \frac{2}{hk^2}$ and hence $\|v_{k+1} - v_k\| \leq \frac{4}{hk^2}$.

Now let $s \in C$ and let $s_{>N}$ denote the projection of s on $S_{N+1} \times S_{N+2} \times \cdots$. We get

$$\|sv_{k+1} - v_{k+1}\| = \frac{\frac{k-2}{k}(sv_k - v_k) + \frac{2}{k}p_{\leq N}(s_{>N}v_k - v_k)}{\left\|\left(\frac{k-2}{k} + \frac{2}{k}p_{\leq N}\right)v_k\right\|}$$

$$
\begin{aligned}
&\leq \frac{\frac{k-2}{k^2} + \frac{2}{k^2(k+1)}}{1 - \frac{2}{hk^2}} \\
&= \frac{k - 2 + \frac{2}{k+1}}{k^2 - 2h^{-1}} \\
&= \frac{k^2 - k}{k^2 - 2h^{-1}} \cdot \frac{1}{k+1} \\
&\leq \frac{1}{k+1}.
\end{aligned}
$$

So v_{k+1} satisfies the desired inequalities.

Now the v_k form a Cauchy sequence, so there is a limit v, and this will be an invariant vector. □

Remark 5.6.3. We could also consider the product topology on G in this Theorem, which is weaker than τ, because the representation constructed in the first part of the proof is also continuous for this topology.

If we take all possible finite quotients (up to isomorphism) of Γ (of which there are necessarily only countably many), we find a new criterion for property (τ).

Corollary 5.6.4. *Let Γ be a finitely generated residually finite group, generated by the symmetric set S. Let $\Gamma_1, \Gamma_2, \dots$ be all finite quotients of Γ and let G be their bounded product. Then Γ has property (τ) if and only if G has topological property (T).*

Proof. It is well known that Γ has property (τ) if and only if the sequences of Cayley graphs $\mathrm{Cay}(\Gamma_n, \overline{S})$ is an expander sequence [33]. By Theorem 5.G, this is equivalent to G having topological property (T). □

Now we turn to the coarse property (T) for the bounded product G. For a sequence of finitely generated groups (Γ_n, S_n) and an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we denote by $\prod_n \Gamma_n/\mathcal{U}$ the (bounded) ultraproduct, that is, the quotient of the set of sequences $G = \{(g_n) \in \Gamma_n \mid \sup_n l(g_n) < \infty\}$ by the equivalence relation $(g_n) \sim (h_n)$ if $\forall_{\mathcal{U}} n[g_n = h_n]$. This is again a finitely generated group. If the S_n are bounded in size, we can identify them all with a fixed symmetric set S, and view Γ_n as a quotient $\varphi_n \colon F_S \twoheadrightarrow \Gamma_n$. Then $\prod_n \Gamma_n/\mathcal{U}$ is equal to $F_S/\lim_{\mathcal{U}} \ker(\varphi_n)$, where $\lim_{\mathcal{U}} \ker(\varphi_n) = \{g \in F_S \mid \forall_{\mathcal{U}} n \colon g \in \ker(\varphi_n)\}$. For any quotient Γ of F_S, we say the *spectral gap* of Γ is the largest $\gamma \in \mathbb{R}$ such that the spectrum of the operator $\Delta_S = \sum_{s \in S} 1 - s \in C^*_{\max}(\Gamma)$ is contained in $\{0\} \cup [\gamma, \infty)$.

Before proving Theorem 5.H we need a lemma. It is clear that any finite product of groups with property (T) has property (T) again. The lemma quantifies this. For a finitely generated group Γ, generated by a symmetric set S containing the identity, let $M_S \in \mathbb{C}[\Gamma]$ denote the averaging operator $M_S = \frac{1}{|S|} \sum_{s \in S} s$. Denote by $\sigma_{\max}(M_S)$ the spectrum of M_S in the maximal C^*-algebra $C^*_{\max}(\Gamma)$. Then Γ has property (T) if and only if there is $\varepsilon > 0$ with $\sigma_{\max}(M_S) \subseteq [\varepsilon - 1, 1 - \varepsilon] \cup \{1\}$.

Lemma 5.6.5. *Let Γ_1, Γ_2 be finitely generated groups, with symmetric generating sets S_1 and S_2. Suppose $\sigma_{\max}(M_{S_i}) \subseteq [\varepsilon - 1, 1 - \varepsilon] \cup \{1\}$ for $i \in \{1, 2\}$. Then the group $\Gamma_1 \times \Gamma_2$ is generated by $S_1 \times S_2$ and $\sigma_{\max}(M_{S_1 \times S_2}) \subseteq [\varepsilon - 1, 1 - \varepsilon] \cup \{1\}$.*

Proof. Consider a representation of $\Gamma_1 \times \Gamma_2$ on a Hilbert space $\mathcal{H}$ without constant vectors. Let $v \in \mathcal{H}$. Note that the subspace $\mathcal{H}^{\Gamma_1}$ of Γ_1-invariant vectors is a subrepresentation, and it contains no Γ_2-invariant vectors. Let w be the projection of v onto $\mathcal{H}^{\Gamma_1}$. Then $\|M_{S_1 \times S_2} w\| = \|M_{S_2} M_{S_1} w\| = \|M_{S_2} w\| \leq (1-\varepsilon)\|w\|$. We also have $\left\|M_{S_1 \times S_2}(v-w)\right\| = \left\|M_{S_2} M_{S_1}(v-w)\right\| \leq \left\|M_{S_1}(v-w)\right\| \leq (1-\varepsilon)\|v-w\|$. we get

$$\|M_{S_1 \times S_2} v\|^2 = \|M_{S_1 \times S_2} w\|^2 + \left\|M_{S_1 \times S_2}(v-w)\right\|^2 \leq (1-\varepsilon)^2 \|v\|^2 .$$

This shows that $\sigma_{\max}(M_{S_1 \times S_2}) \subseteq [\varepsilon - 1, 1-\varepsilon] \cup \{1\}$. ☐

We now recall the statement of Theorem 5.H.

Theorem 5.6.6. *Let Γ_n be a sequence of finite groups with symmetric generator sets S_n. Suppose that $\sup_n |S_n| < \infty$. Let X be the coarse disjoint union of the Cayley graphs $\mathrm{Cay}(\Gamma_n, S_n)$. The following are equivalent:*

- *(i) The coarse space X has geometric property (T).*
- *(ii) All ultraproducts $G_{\mathcal{U}} = \prod_n \Gamma_n/\mathcal{U}$ have property (T) with uniform spectral gap.*
- *(iii) All ultraproducts $G_{\mathcal{U}}$ have property (T).*
- *(iv) There is a finitely generated group Γ with property (T) such that all Γ_n are a quotient of Γ.*
- *(v) The bounded product G has coarse property (T).*
- *(vi) The bounded product G has coarse property (T-).*

Proof. We can assume without loss of generality that $|S_n|$ is constant. Then, we can view each Γ_n as a quotient $\varphi_n \colon F_S \twoheadrightarrow \Gamma_n$. The ultralimit of these maps is $\varphi_{\mathcal{U}} \colon F_S \twoheadrightarrow G_{\mathcal{U}}$.

Suppose (i) holds. Let $\mathbb{C}_{\mathrm{cs}}[X]$ denote the Roe algebra of X, and let $\gamma > 0$ such that $\sigma(\pi(\Delta_S)) \subseteq \{0\} \cup [\gamma, \infty)$ for all representations π. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and consider a unitary representation $\pi \colon G_{\mathcal{U}} \to U(\mathcal{H})$. Suppose there is a unit vector $v \in \mathcal{H}$, and η with $\pi(\Delta_S)v = \eta v$. We will show that $\eta \in \{0\} \cup [\gamma, \infty)$, which implies that $G_{\mathcal{U}}$ has property (T) with spectral gap at least γ.

For any n, consider the Hilbert space $\ell^2(X_n, \mathcal{H})$. The group F_S acts on this by $g \cdot \xi(x) = \pi(g)(\xi(g^{-1}x))$ for $\xi \in \ell^2(X_n, \mathcal{H})$ and $x \in X_n$. Now let $\mathcal{H}' = \prod_n \ell^2(X_n, \mathcal{H})/\mathcal{U}$. We define a representation $\pi' \colon \mathbb{C}_{\mathrm{cs}}[X] \to B(\mathcal{H}')$ as follows. Any element of $\mathbb{C}_{\mathrm{cs}}[X]$ can (usually non-uniquely) be written as a finite sum $\sum_{g \in F_S} f_g g$ with $f_g \in \ell^\infty X$. In other words, $\mathbb{C}_{\mathrm{cs}}[X]$ is a quotient of the $*$-algebra $\ell^\infty(X) \rtimes F_S$. We define π' by

$$\pi'\left(\sum_{g \in F_S} f_g g\right)((\xi_n)) = \left(\left(\sum_{g \in F_S} f_g \cdot (g \cdot \xi_n)\right)\right)$$

for $((\xi_n)) \in \prod_n \ell^2(X_n, \mathcal{H})/\mathcal{U}$. This is a well-defined $*$-representation of $\ell^\infty(X) \rtimes F_S$. To see that it induces a representation of $\mathbb{C}_{\mathrm{cs}}[X]$, suppose that $\sum_{g \in F_S} f_g g = 0$ in $\mathbb{C}_{\mathrm{cs}}[X]$. For any $g, h \in F_S$ we know that if $\varphi_n(g) = \varphi_n(h)$ modulo $\mathcal{U}$, then $\pi(g) = \pi(h)$. Since only finitely many g occur in the sum, we know that there is $A \in \mathcal{U}$ such that for all $n \in A$ and all g, h with $f_g \neq 0$ and $f_h \neq 0$, we have that $\varphi_n(g) = \varphi_n(h)$ implies $\pi(g) = \pi(h)$. Now for $n \in A$ and $x \in X_n$ we have

$$\sum_{g \in F_S} f_g \cdot (g \cdot \xi_n)(x) = \sum_{y \in X_n} \sum_{g | gy = x} f_g(x) \pi(g)(\xi_n(y)).$$

Now if g, h both satisfy $gy = x$ and $hy = x$ we have $\varphi_n(g) = \varphi_n(h)$, so $\pi(g) = \pi(h)$. Moreover by the assumption that $\sum_{g \in F_S} f_g g = 0$ we know that $\sum_{g | gy = x} f_g(x) = 0$ for all $x, y \in X_n$, we see that the inner sum is zero for each y. Therefore, the map π' is well-defined representation of $\mathbb{C}_{\mathrm{cs}}[X]$.

Consider the unit vector $((\xi_n)) \in \mathcal{H}'$ given by $\xi_n(x) = |X_n|^{-\frac{1}{2}} v$ for all $x \in X_n$. Then $\pi'(\Delta_S)((\xi_n)) = \eta((\xi_n))$. This shows that $\eta \in \sigma(\pi'(\Delta_S)) \subseteq \{0\} \cup [\gamma, \infty)$, as we wanted to prove.

The implication (ii) $\Longrightarrow$ (iii) is trivial. Now suppose (iii). First we prove that there is an integer R such that the group $F_S / \langle \ker(\varphi_n) \cap S^R \rangle$ has property (T), for all n. If this is not the case, then for every R there is an $n_R \in \mathbb{N}$ such that $F_S / \langle \ker(\varphi_{n_R}) \cap S^R \rangle$ does not have property (T). Let $\mathcal{U}$ be a non-principal ultrafilter and consider the ultraproduct $H = \prod_R \Gamma_{n_R} / \mathcal{U}$. This (discrete) group has property (T) by assumption. By the main theorem of [41], we can write

$$\Delta_S^2 - \gamma \Delta_S = \sum_{i=1}^{k} \xi_i^* \xi_i \in \mathbb{R}[H]$$

for some $\gamma > 0$ and $\xi_i \in \mathbb{R}[H]$. Let $\tilde{\xi_i}$ be lifts of ξ_i in $\mathbb{R}[F_S]$. For large enough R, the supports of the $\tilde{\xi_i}$ will be contained in $S^{R/4}$. Now for most R (in the sense of the ultrafilter) we have

$$\Delta_S^2 - \gamma \Delta_S - \sum_{i=1}^{k} \xi_i^* \xi_i \in \ker(\mathbb{R}[F_S] \twoheadrightarrow \mathbb{R}[\Gamma_{n_R}]) \cap \mathbb{R}[S^{R/2}] \subseteq \ker(\mathbb{R}[F_S] \twoheadrightarrow \mathbb{R}[F_S / \langle \ker(\varphi_{n_R}) \cap S^R \rangle]).$$

So we see that for these R the spectral gap of $F_S / \langle \ker(\varphi_{n_R}) \cap S^R \rangle$ is at least γ, giving a contradiction.

So let R be such that $F_S / \langle \ker(\varphi_n) \cap S^R \rangle$ has property (T) for all n. Let $\Lambda_1, \ldots, \Lambda_m$ be all the groups that can arise in this way. There are only finitely many since there are only finitely many subsets of S^R. All these groups have property (T). Then their product Γ has property (T), and all Γ_n are a quotient of Γ, showing the implication (iii) $\Longrightarrow$ (iv).

Now suppose (iv) is true, so all Γ_n are quotients of a single group Γ with property (T). We can also assume that Γ is generated by S and S_n is the image of S. We will show that G has coarse property FH, which is equivalent to coarse property (T) since the bornology of G is generated by the bounded set $\prod_n S_n$.

We can assume that S is symmetric and contains the identity. Then there is $\varepsilon > 0$ such that $\sigma_{\max}(M_S) \subseteq [\varepsilon - 1, 1 - \varepsilon] \cup \{1\}$. Let $\alpha \colon G \curvearrowright \mathcal{H}$ be a controlled affine action. Let $v \in \mathcal{H}$ and let $C > 0$ such that $\|\alpha(s)v - v\| \leq C$ for all $s \in \prod_n S_n$. Let $g \in G$. We will show that $\|\alpha(g)v - v\| \leq 2C/\varepsilon$, showing that α has bounded orbits.

Since g is in the bounded product, there is an integer k such that $g \in (\prod_n S_n)^k$. Each coordinate of g is then an image of an element in S^k. Since S^k is finite, this means there is an integer m and a homomorphism $f \colon \Gamma^m \to G$ and an element $x \in \Gamma^m$ with $f(x) = g$. By Lemma 5.6.5, we have $\sigma_{\max}(M_{S^m}) \subseteq [\varepsilon - 1, 1 - \varepsilon] \cup \{1\}$.

Let v_0 denote the orthogonal projection of v on the subspace of $\mathcal{H}$ consisting of Γ^m-fixed vectors. Define $\pi \colon \Gamma^m \curvearrowright \mathcal{H}$ by $\pi(y)w = \alpha(f(y))(w + v_0) - v_0$. This defines a representation of π on $\mathcal{H}$. Since $v - v_0$ is perpendicular to the constant vectors of this representation, we have $\|\pi(M_{S^n})(v - v_0)\| \leq (1 - \varepsilon)\|v - v_0\|$. We also have

$$\|\pi(M_{S^n})(v - v_0)\| \geq \|v - v_0\| - \left\| \frac{1}{|S^m|} \sum_{s \in S^m} \alpha(f(s))v - v \right\| \geq \|v - v_0\| - C.$$

This gives $\|v - v_0\| \le C/\varepsilon$. Finally we have

$$\big\|\alpha(g)v - v\big\| = \big\|\alpha(f(x))v - v\big\| = \big\|\pi(x)(v - v_0) + v_0 - v\big\| \le 2\|v_0 - v\| \le 2C/\varepsilon.$$

This shows that α has bounded orbits, thus G has coarse property FH, and coarse property (T) as well, showing (iv) $\Longrightarrow$ (v).

Suppose G has coarse property (T). We will prove that X has geometric property (T). Let (A, ε) be a Kazhdan pair for G. Let $N = \sup_{(g_n)\in A} \sup_n l(g_n)$ which is finite since A is bounded. Let $\pi\colon \mathbb{C}_{\mathrm{cs}}[X] \to B(\mathcal{H})$ be a representation. Let $v \in \mathcal{H}_c^\perp$ be a unit vector, perpendicular to all constant vectors. Note that G acts on L^2X, and this gives a natural embedding $G \to \mathbb{C}_{\mathrm{cs}}[X]$. Thus π restricts to a representation of G. The constant vectors form a subrepresentation, so $\mathcal{H}_c^\perp$ is also a representation of G. It does not have any constant vectors. So there is $a \in A$ with $\|av - v\| > \varepsilon$. Now a can be viewed as an element in $\mathbb{C}_{cs}[X]$ with propagation at most N. So it follows that X has geometric property (T), showing (v) $\Longrightarrow$ (i).

The implication (v) $\Longrightarrow$ (vi) is trivial. We will finish by showing (vi) $\Longrightarrow$ (iii). Suppose G has property (T-), or equivalently FH-, and let $\alpha\colon G_{\mathcal{U}} \curvearrowright \mathcal{H}$ be an affine action. Then the composition $G \to G_{\mathcal{U}} \to U(\mathcal{H})$ gives a well-controlled action, because the image of any bounded subset of G is finite. This shows that a fixed point exists, so $G_{\mathcal{U}}$ has property FH and property (T). $\square$

5.7 Further remarks and questions

Cohomological interpretation. By the Mazur-Ulam theorem, an affine isometric action of G on a Banach space E corresponds a pair (π, b), where π is a norm-preserving representation π of G on E, and $b \in Z^1(G, \pi)$ is 1-cocycle, i.e. a map $b : G \to E$ satisfying the cocycle relation $b(gh) = \pi(g)b(h) + b(g)$ for all $g, h \in G$. The subspace $B^1(G, \pi)$ of $Z^1(G, \pi)$ consists of coboundaries, i.e. cocycles of the form $b(g) = v - \pi(g)v$ where $v \in E$ (this corresponds to the affine action fixing the point v). The first cohomology groups with values in π is defined as $H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi)$.

Assume that G is a bornological group. Note that an affine action given as $\sigma(g)v = \pi(g)v + b(g)$. Hence σ is controlled if and only if b is "controlled": i.e. bounded on bounded sets. The subspaces of controlled cocycles will be denoted by $Z^1_{\mathrm{born}}(G, \pi)$. Note that $B^1(G, \pi) \subset Z^1_{\mathrm{born}}(G, \pi)$. The *controlled first cohomology* will be defined as $H^1_{\mathrm{born}}(G, \pi) = Z^1_{\mathrm{born}}(G, \pi)/B^1(G, \pi)$.

If G is a topological group, and π is a norm-preserving representation (not necessarily continuous), we let $Z^1_{\mathrm{cont}}(G, \pi)$ be the subspace of continuous 1-cocycles, and let $H^1_{\mathrm{cont}}(G, \pi) = Z^1_{\mathrm{cont}}(G, \pi)/B^1(G, \pi)$. If G is locally compact, equipped with the bornology $\mathcal{K}$, then the inclusion $Z^1_{\mathrm{cont}}(G, \pi) \to Z^1_{\mathrm{born}}(G, \pi)$ gives rise to an injective linear map $H^1_{\mathrm{cont}}(G, \pi) \to H^1_{\mathrm{born}}(G, \pi)$.

Theorem 5.7.1. *Let G be a locally compact group. Let π be a norm-preserving (not necessarily continuous) representation on a uniformly convex Banach space E. Then every controlled 1-cocycle is cohomologous to a continuous one. In other words, the map $H^1_{\mathrm{cont}}(G, \pi) \to H^1_{\mathrm{born}}(G, \pi)$ is an isomorphism.*

Proof. Let $b \in Z^1_{\mathrm{born}}(G, \pi)$, and let σ be the corresponding (controlled) affine action. Proposition 5.1.5 implies that there is $v \in E$ such that $g \mapsto \sigma(g) \cdot v$ is continuous. Let c be the 1-coboundary defined by $c(g) = \pi(g)v - v$. Then the cocycle $b' = b + c$ satisfies $b'(g) = b(g) + \pi(g)v - v = \sigma(g) \cdot v - v$. It is therefore continuous. $\square$

Note that the previous definitions extend to higher degree cohomology groups. In particular, we have a injective map $H^k_{\mathrm{cont}}(G, \pi) \to H^k_{\mathrm{born}}(G, \pi)$.

Question 5.7.2. *Under the assumptions of Theorem 5.7.1, is the map $H^k_{\mathrm{cont}}(G, \pi) \to H^k_{\mathrm{born}}(G, \pi)$ surjective (hence bijective) for all $k \geq 1$?*

Shopping centres for Banach spaces. The proofs of Theorems 5.F and 5.D rely on the notion of shopping centre (see §5.2.3) which attach to every bounded subset of a Hilbert space a relatively compact subset which satisfies a crucial property stated in Lemma 5.2.11. We have not been able to define an analogous notion for uniformly convex Banach spaces.

Question 5.7.3. *Is there an analogous version of Lemma 5.2.11 for uniformly convex Banach spaces?*

Question 5.7.4. *Are Theorems 5.D and 5.F valid for uniformly convex Banach spaces?*

Bounded products. The notion of bounded product makes sense for a family of groups equipped with a left-invariant pseudo-metric. It would be interesting to investigate it more systematically (see also [54]). In particular:

Question 5.7.5. *Can Theorems 5.G and 5.H have analogues for sequences of compact groups equipped with suitable left-invariant metrics?*

Stability of coarse FH. Among locally compact groups, property FH is invariant under taking closed cocompact unimodular subgroups. In particular this holds if the subgroup is closed cocompact and normal. The unimodulaityr assumption is crucial as for example, $\mathrm{SL}_3(\mathbb{R})$ has property FH, but the closed cocompact subgroup of upper-triangular matrices does not (as it is solvable). The situation is even worse among Polish groups: indeed we have seen that $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ has (even coarse) property FH but contains $\mathbb{Z}$ as a coarsely dense discrete subgroup. Note that for locally compact groups, the unimodularity assumption of the subgroup $H < G$ is here to ensure that G/H have a G-invariant probability measure.

Question 5.7.6. *Is there a reasonably natural restriction on pairs (G, H), where G is a bornological group and H is a subgroup which ensures that property FH for G implies property FH for H?*

6 Actions of amenable groups on measure spaces

Chapter summary

We show that every *controlled* action of a locally compact amenable group on a finitely-additive measure space has an invariant mean. This gives a characterisation of amenability that does not use the Haar measure. In fact, it only relies on the bornological structure of the group, making it an excellent starting point for the systematic study of amenability for bornological groups.

6.1 Introduction and main results

Recall that a locally compact group G is amenable if there exists an invariant mean on $L^\infty G$. In this chapter we consider a locally compact group G with a measure-preserving action on a finitely-additive measure space (X, ν). We do not require that this action is continuous or even measurable (it is also not clear what continuity would mean in this context). Instead we just require that the action is *controlled*.

Definition 6.1.1. Let G be a locally compact topological group and let (X, ν) be a finitely-additive measure space. A measure-preserving action $\alpha\colon G \curvearrowright X$ is called *controlled* if there is a measurable subset $A \subseteq X$ with $0 < \nu(A) < \infty$ such that the following condition is satisfied: for every compact subset $K \subseteq G$ and all $\varepsilon > 0$ there is a measurable subset $B \subseteq X$ with $\nu(B) < \infty$ and with $\nu(gA \cap B) \geq (1-\varepsilon)\nu(A)$ for all $g \in K$.

Example 6.1.2. For every locally compact group G with left-invariant Haar measure μ, the action of G on (G, μ) by left-multiplication is controlled. We can take A to be any compact neighbourhood of the identity, and for a compact $K \subseteq G$, let $B = KA$, which is still compact, so it has finite measure.

Definition 6.1.3. Let G be a locally compact topological group and (X, ν) be a finitely-additive measure space. Let $\alpha\colon G \curvearrowright X$ be a measure-preserving action. An *invariant mean* on X is a finitely-additive measure φ on the same algebra as ν satisfying $\varphi(X) = 1$ and $\varphi(gA) = \varphi(A)$ for all measurable A and $g \in G$.

Theorem 6.A. Let G be a locally compact topological group. Then G is amenable if and only if for all controlled measure-preserving actions $G \curvearrowright (X, \nu)$ on a finitely-additive measure space there is an invariant mean on X.

Interestingly, this gives a characterisation of amenability that does not use the Haar measure. It also does not require continuity of the action.

We will prove Theorem 6.A first for a σ-additive measure ν. For this the idea is to choose a vector in L^2X, take the orbit under a Følner set in G and take the average. This should give a vector in L^2X with good invariance properties, and taking a limit will give an invariant mean. The challenge is that the action of G on L^2X has no regularity properties like continuity or measurability, so we can not straightforwardly take this kind of average. We introduce the L^1-mean centre to tackle this problem.

Then we extend the result to finitely-additive measure spaces by taking an ultraproduct of X and equipping it with the *Loeb measure*. This is a σ-additive measure, even if the original measure was only finitely additive. Then we can apply the case of σ-additive measures to get an invariant mean on the ultraproduct and we show that from this we can construct an invariant mean on X.

6.2 L^1-mean centre

We introduce the L^1-mean centre. This construction is similar to the construction of the mean centre in Section 5.2, but the mean is taken in a different way. Because there is an interplay between L^1X and L^2X in the construction, we need to require the subset that we take the mean of to consist of positive functions. For a σ-additive measure space X, we denote by $(L^pX)^+$ the cone consisting of all functions $f \in L^pX$ with $f \geq 0$.

Construction 6.2.1. *Let (X, ν) be a measure space. Let $C \subseteq L^2X^+$ be a bounded subset. Let Σ be an algebra on C and $\mu\colon \Sigma \to [0,\infty)$ a finitely additive function with $\mu(C) > 0$. Then we construct the* L^1-mean centre *$\tilde{Z}(\mu) \in L^2(X,\nu)^+$ satisfying the following property:*

If $A \in \Sigma$ then

$$\tilde{Z}(\mu)^2 = \frac{\mu(A)}{\mu(C)}\tilde{Z}(\mu_{|A})^2 + \frac{\mu(C \setminus A)}{\mu(C)}\tilde{Z}(\mu_{|C\setminus A})^2.$$

Proof. Recall from Section 5.2 that a non-empty bounded subset $A \subseteq \mathcal{H}$ of a Hilbert space has a unique centre $Z(A)$ and a radius $\rho(A) = \sup_{a\in A}\big\|a - Z(A)\big\|$. Let $s = \sup_{x\in C}\|x\|_2$. Let $P = (A_i)$ be a finite partition of C in elements of Σ. We define

$$\tilde{Z}_P(\mu) = \left(\sum_i \frac{\mu(A_i)}{\mu(C)} Z(A_i)^2\right)^{\frac{1}{2}} \in L^2(X,\nu)^+,$$

$$\tilde{\rho}_P(\mu) = \left(\sum_i \frac{\mu(A_i)}{\mu(C)} \rho(A_i)^2\right)^{\frac{1}{2}} \in [0,\infty).$$

We want to take a limit as P gets more refined, so we need to prove an inequality to show convergence. Let $P = (A_i)$ be a finite partition of C in elements of Σ, and let $Q = (B_{ij})$ be a refinement. Then we have, using the Cauchy-Schwarz inequality and Remark 5.2.4:

$$\begin{aligned}
\Big\|\tilde{Z}_P(\mu) - \tilde{Z}_Q(\mu)\Big\|_2^2 &\leq \Big\|\tilde{Z}_P(\mu)^2 - \tilde{Z}_Q(\mu)^2\Big\|_1 \\
&= \frac{1}{\mu(C)}\left\|\sum_i\left(\sum_j \mu(B_{ij})(Z(A_i)^2 - Z(B_{ij})^2)\right)\right\|_1 \\
&\leq \frac{1}{\mu(C)}\sum_{i,j}\mu(B_{ij})\Big\|Z(A_i)^2 - Z(B_{ij})^2\Big\|_1 \\
&\leq \frac{1}{\mu(C)}\sum_{i,j}\mu(B_{ij})\big\|Z(A_i) - Z(B_{ij})\big\|_2 \cdot \big\|Z(A_i) + Z(B_{ij})\big\|_2 \\
&\leq \frac{2s}{\mu(C)}\sum_{i,j}\mu(B_{ij})\big\|Z(A_i) - Z(B_{ij})\big\|_2 \\
&\leq \frac{2s}{\mu(C)^{\frac{1}{2}}}\left(\sum_{i,j}\mu(B_{ij})\big\|Z(A_i) - Z(B_{ij})\big\|_2^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2s}{\mu(C)^{\frac{1}{2}}}\left(\sum_{i,j}\mu(B_{ij})(\rho(A_i)^2-\rho(B_{ij})^2)\right)^{\frac{1}{2}}\\
&= 2s(\tilde{\rho}_P(\mu)^2-\tilde{\rho}_Q(\mu)^2)^{\frac{1}{2}}.
\end{aligned}
$$

In particular we see that $\tilde{\rho}_P(\mu) \geq \tilde{\rho}_Q(\mu)$. Now let P_n be a sequence of finite partitions of C in elements of Σ, each a refinement of the next, such that $\tilde{\rho}_{P_n}(\mu)$ converges to the infimum of $\tilde{\rho}_P(\mu)$ taken over all finite partitions P of C in elements of Σ. Then the above computation shows that $\tilde{Z}_P(\mu)$ converges to an element $\tilde{Z}(\mu)$, that is not dependent on the chosen sequence of partitions.

Now let $A \in \Sigma$. We may take the P_n to be refinements of the partition $\{A, C\setminus A\}$. Then these partitions split in a partition $P_{n,1}$ of A and a partition $P_{n,2}$ of $C\setminus A$, and we get

$$\tilde{Z}_{P_n}(\mu)^2 = \frac{\mu(A)}{\mu(C)}\tilde{Z}_{P_{n,1}}(\mu_{|A})^2 + \frac{\mu(C\setminus A)}{\mu(C)}\tilde{Z}_{P_{n,2}}(\mu_{|C\setminus A})^2$$

and

$$\tilde{\rho}_{P_n}(\mu)^2 = \frac{\mu(A)}{\mu(C)}\tilde{\rho}_{P_{n,1}}(\mu_{|A})^2 + \frac{\mu(C\setminus A)}{\mu(C)}\tilde{\rho}_{P_{n,2}}(\mu_{|C\setminus A})^2.$$

Taking the limit as $n\to\infty$ we get

$$\tilde{Z}(\mu)^2 = \frac{\mu(A)}{\mu(C)}\tilde{Z}(\mu_{|A})^2 + \frac{\mu(C\setminus A)}{\mu(C)}\tilde{Z}(\mu_{C\setminus A})^2.$$

□

6.3 The σ-additive case

Consider a measure-preserving action of a locally compact group on a σ-additive measure space. As usual, there are multiple characterisations of the existence of an invariant mean.

Proposition 6.3.1. *Let G be a group and $G \curvearrowright (X,\nu)$ be a measure-preserving action. The following are equivalent:*

(i) There is a G-invariant mean on X.

(ii) There are almost G-invariant vectors in L^2X.

(iii) For every finite $F\subseteq G$ and $\varepsilon>0$, there is a measurable $A\subseteq X$ of finite measure with $\nu(g\cdot A\setminus A)<\varepsilon\nu(A)$ for all $g\in F$.

Proof. We start by proving (i) $\Longrightarrow$ (iii). Assume (i) and let φ be a G-invariant mean on X. Let $F\subseteq G$ be a symmetric finite subset and let $\varepsilon>0$. Assume for a contradiction that for every measurable $A\subseteq X$ with finite measure we have $\nu(F\cdot A\setminus A)\geq\varepsilon\nu(A)$.

Let $\mathcal{P}(X)=\{\psi\in L^1X \mid \psi\geq 0, \int\psi=1\}$. Note that, by the Goldstine theorem, φ is in the weak closure of $\mathcal{P}(X)$. Consider the Banach space $\mathcal{B}=\sum_{g\in F}L^1X$, with the norm $\|(\psi_g)\|=\sum_g\|\psi_g\|_1$. The set

$$\{(\psi-g^*\psi)\in\mathcal{B}\mid\psi\in\mathcal{P}(X)\}$$

is convex. If $\psi \to \varphi$ weakly, then $\psi - g^*\psi \to 0$ weakly, since φ is G-invariant. So 0 is in the weak closure of this set. Since it is convex, it is also in the norm closure. We can then pick $\psi \in \mathcal{P}(X)$ with $\sum_{g\in F}\|\psi - g^*\psi\|_1 \leq \eta$, where we will choose $\eta > 0$ later. We also pick a fixed representative function for ψ, which we also denote by ψ.

Let $(1+\varepsilon)^{-1} < \delta < 1$. For $n \in \mathbb{Z}$, let

$$A_n = \{x \in X \mid \psi(x) \geq \delta^n\}.$$

Clearly, $\nu(A_n) < \infty$ so $\nu(FA_n \backslash A_n) \geq \varepsilon\nu(A_n)$ for all n. Let $D_n = FA_n \backslash A_{n+1}$. For each $x \in X$ there are n_0, n_1 such that x is contained in $D_{n_0}, D_{n_0+1}, \ldots, D_{n_1}$ and no other D_n. Then we have $\psi(gx) \geq \delta^{n_0}$ for some $g \in F$, and $\psi(x) < \delta^{n_1+1}$. So $\sum_g |\psi(fx) - \psi(x)| \geq \delta^{n_0} - \delta^{n_1+1} = \sum_{n=n_0}^{n_1} \delta^n - \delta^{n+1}$. We get

$$\eta \geq \sum_g \|\psi - g^*\psi\|_1 \geq \sum_n (\delta^n - \delta^{n+1})\nu(D_n) = (1-\delta)\sum_n \delta^n \nu(D_n).$$

We also have

$$(1+\varepsilon)\nu(A_n) \leq \nu(FA_n) \leq \nu(A_{n+1}) + \nu(D_n).$$

Multiplying by δ and summing over all n gives

$$(1+\varepsilon)\sum_n \delta^n \nu(A_n) \leq \sum_n \delta^n \nu(A_{n+1}) + \sum_n \delta^n \nu(D_n).$$

So

$$(1+\varepsilon-\delta^{-1})\sum_n \delta^n \nu(A_n) \leq \sum_n \delta^n \nu(D_n) \leq (1-\delta)^{-1}\eta.$$

On the other hand, we have

$$1 = \sum_n \int_{A_{n+1}\backslash A_n} \psi \leq \sum_n (\nu(A_{n+1}) - \nu(A_n))\delta^n = (\delta^{-1}-1)\sum_n \delta^n \nu(A_n) \leq \delta^{-1}(1+\varepsilon-\delta^{-1})^{-1}\eta.$$

Picking η small enough gives a contradiction.

The implication (iii) $\Longrightarrow$ (ii) is straight-forward: let A be a measurable set with finite measure and $\nu(g \cdot A \setminus A) < \varepsilon\nu(A)$ for all $g \in F$. Let $\xi = \nu(A)^{-1/2}\mathbb{1}_A$ be the normalized characteristic function of A. Then $\|\xi - g^*\xi\| = \nu(A)^{-1/2}\nu(A\Delta gA)^{1/2} < \sqrt{2\varepsilon}$.

Lastly, we prove (ii) $\Longrightarrow$ (i), so we assume that there are almost G-invariant vectors in L^2G. Let

$$\Lambda = \{F \subseteq G \text{ finite }, \varepsilon > 0\}.$$

This is a directed set where $(F,\varepsilon) \leq (F',\varepsilon')$ if $F' \subseteq F$ and $\varepsilon \geq \varepsilon'$. For $\lambda = (F,\varepsilon) \in \Lambda$ let $\xi_\lambda \in L^2G$ be a unit vector such that $\|\xi_\lambda - g^*\xi_\lambda\| < \varepsilon$ for all $g \in F$. We can assume that the ξ_λ are real-valued. Let $\varphi_\lambda \in (L^\infty X)^*$ be the unit vector given by $\varphi_\lambda(h) = \int_X h(x)\xi_\lambda(x)^2 d\nu(x)$. Then for $g \in F^{-1}$ we have

$$\varphi_\lambda(f^*h) - \varphi_\lambda(h) = \int_X h(x)(\xi_\lambda(gx)^2 - \xi_\lambda(x)^2)d\nu(x)$$

so

$$|\varphi_\lambda(g^*h) - \varphi_\lambda(h)| \leq \|h\| \cdot \int_X |\xi_\lambda(gx)^2 - \xi_\lambda(x)^2| d\nu(x)$$

$$
\begin{aligned}
&\leq \left(\int_X \left|\xi_\lambda(gx)+\xi_\lambda(x)\right|^2 d\nu(x)\right)^{\frac{1}{2}} \cdot \left(\int_X \left|\xi_\lambda(gx)-\xi_\lambda(x)\right|^2 d\nu(x)\right)^{\frac{1}{2}}\\
&\leq 2^{\frac{1}{2}} \left\|(g^{-1})^*\xi_\lambda - \xi_\lambda\right\|^{\frac{1}{2}}\\
&\leq (2\varepsilon)^{\frac{1}{2}}.
\end{aligned}
$$

So $\|g\varphi_\lambda - \varphi_\lambda\| \leq (2\varepsilon)^{\frac{1}{2}}$. The unit ball in $(L^\infty X)^*$ is compact, so the net φ_λ has a limit point φ. Then φ is an invariant mean. $\square$

Proposition 6.3.2. *Let G be a locally compact topological group. Then G is amenable if and only if for all controlled measure-preserving actions $G \curvearrowright (X,\nu)$ on a σ-additive measure space there is an invariant mean on X.*

Proof. One direction follows from Example 6.1.2. For the other direction, suppose that G is amenable and let $\alpha\colon G \curvearrowright (X,\nu)$ be a controlled action. Let $A \subseteq X$ be as in Definition 6.1.1. Let $f\colon G \to L^2X$ be the function $f(g) = \mathbb{1}_{gA}$. Let $C \subseteq G$ be a fixed compact neighbourhood of the identity and let $B \subseteq X$ be such that $\nu(gA \cap B) \geq \frac{1}{2}\nu(A)$ for all $g \in C$.

Let $\varepsilon > 0$ and $F \subseteq G$ a finite set. There is a compact Kazhdan set $K \subseteq G$ satisfying $\mu(K\Delta gK) < \varepsilon\mu(K)$ for all $g \in F$. Let $\eta = \eta_{F,\varepsilon} = \tilde{Z}((f_{|K})_*\mu)^2 \in L^1(X,\nu)^+$. We will show that η is almost invariant for F. First we will find a lower bound for $\|\eta\|_1$.

Since K is compact, we may write K as a finite partition $K = \bigsqcup_i g_iC_i$, with $g_i \in G$ and $C_i \subseteq C$ measurable. Consider any partition $\mathcal{P} = (A_{ij})$ subordinate to this one. For any $a \in A_{ij}$ we have $\int_{g_iB} f(a)d\nu = \nu(aA \cap g_iB) = \nu(g_i^{-1}aA \cap B) \geq \frac{1}{2}\nu(A)$. Since $Z(f(A_{ij})) \in \overline{\mathrm{co}}(f(A_{ij}))$ we get $\int_{g_iB} Z(f(A_{ij}))d\nu \geq \frac{1}{2}\nu(A)$ as well. From this we conclude that $\left\|Z(f(A_{ij}))\right\|_2^2 \geq \frac{1}{4}\nu(A)^2\nu(B)^{-1}$. Since

$$
\tilde{Z}_{\mathcal{P}}((f_{|K})_*\mu)^2 = \sum_{ij} \frac{\mu(A_{ij})}{\mu(K)} \left\|Z(f(A_{ij}))\right\|_2^2,
$$

we get

$$
\left\|\tilde{Z}_{\mathcal{P}}((f_{|K})_*\mu)\right\|_2^2 \geq \frac{1}{4}\nu(A)^2\nu(B)^{-1}.
$$

Taking the limit as $\mathcal{P}$ gets more refined, we conclude that $\|\eta\|_1 \geq \frac{1}{4}\nu(A)^2\nu(B)^{-1}$.

Now let $g \in F$. We have $\alpha_g(\eta) = \tilde{Z}((\alpha_g)_*(f_{|K})_*\mu)^2 = \tilde{Z}((f_{|gK})_*\mu)^2$. Then we get

$$
\eta = \frac{\mu(gK \cap K)}{\mu(K)} \tilde{Z}((f_{gK\cap K})_*\mu)^2 + \frac{\mu(K \setminus gK)}{\mu(K)} \tilde{Z}((f_{K\setminus gK})_*\mu)^2
$$

and

$$
\alpha_g(\eta) = \frac{\mu(gK \cap K)}{\mu(K)} \tilde{Z}((f_{gK\cap K})_*\mu)^2 + \frac{\mu(gK \setminus K)}{\mu(K)} \tilde{Z}((f_{gK\setminus K})_*\mu)^2.
$$

We find

$$
\alpha_g(\eta) - \eta = \frac{\mu(gK \setminus K)}{\mu(K)} (\tilde{Z}((f_{|gK\setminus K})_*\mu)^2 - \tilde{Z}((f_{K\setminus gK})_*\mu)^2)
$$

so $\left\|\alpha_g(\eta) - \eta\right\|_1 \leq \frac{2\mu(gK\setminus K)\nu(A)}{\mu(K)} \leq \varepsilon\nu(A)$.

Now let $\zeta_{F,\varepsilon} = \frac{\eta_{F,\varepsilon}}{\|\eta_{F,\varepsilon}\|_1}$ be the normalisation of η. Then by the estimates above, we get that $\|\alpha_g(\zeta_{F,\varepsilon}) - \zeta_{F,\varepsilon}\|_1 \leq 4\varepsilon\nu(B)\nu(A)^{-1}$ for all $g \in F$. Since $\zeta_{F,\varepsilon} \in L^1(X,\nu)^+$ has integral one it defines a mean on X. Let φ be a weak limit point of these means as F increases to G and ε decreases to zero, then φ is an invariant mean on X. $\square$

6.4 The Loeb measure

An important construction is that of the *Loeb measure* on an ultraproduct of measure spaces, as described in [31] and [52], for example. We describe the construction in detail for ultraproducts indexed on arbitrary directed sets, rather than just on the natural numbers.

Let $(\Lambda, \leq)$ be a directed set without maximum. By Lemma 5.4.3, there is a cofinal ultrafilter $\mathcal{U}$ on Λ and a net n_λ with values in the natural numbers, such that $\lim_{\mathcal{U}} n_\lambda = \infty$. Suppose for each $\lambda \in \Lambda$ we have a space X_λ, an algebra Σ_λ of subsets of X_λ, and a finitely additive function ν_λ on Σ_λ. Consider the ultraproduct $X_{\mathcal{U}} = \prod_\lambda X_\lambda/\mathcal{U}$.

Suppose we have subsets $B_\lambda \subseteq X_\lambda$ for every λ. We denote by $\lim_{\mathcal{U}} B_\lambda \subseteq X_{\mathcal{U}}$ the subset consisting of equivalence classes (x_λ) of sequences with $x_\lambda \in B_\lambda$ for most λ (with respect to $\mathcal{U}$). Such a set is called a *quasibox*. Note that $\lim_{\mathcal{U}}(B_\lambda \cap B'_\lambda) = \lim_{\mathcal{U}} B_\lambda \cap \lim_{\mathcal{U}} B'_\lambda$ and $X_{\mathcal{U}} \setminus \lim_{\mathcal{U}} B_\lambda = \lim_{\mathcal{U}}(X_\lambda \setminus B_\lambda)$. Therefore the set of quasiboxes forms an algebra $\mathcal{A}$ of subsets of $X_{\mathcal{U}}$. We can define the finitely additive function ν on $\mathcal{A}$ by

$$\nu(\lim_{\mathcal{U}} B_\lambda) = \lim_{\mathcal{U}} \nu_\lambda(B_\lambda) \in [0, \infty].$$

To show that ν can be extended to a σ-additive measure, we need the following Lemma (see [52, Proposition 1.1]).

Lemma 6.4.1. *Let $B^{(n)}$ be a sequence of quasiboxes in $X_{\mathcal{U}}$.*

(i) If $\bigcap_{n \leq N} B^{(n)} \neq \emptyset$ for all N, then $\bigcap_n B^{(n)} \neq \emptyset$.

(ii) If $\nu(\bigcap_{n \leq N} B^{(n)}) > c$ for all N, then $\bigcap_n B^{(n)}$ contains a quasibox A with $\nu(A) \geq c$.

Proof. Recall there is a net n_λ with $\lim_{\mathcal{U}} n_\lambda = \infty$. Write $B^{(n)} = \lim_{\mathcal{U}} B^{(n)}_\lambda$.

(i) For every $\lambda \in \Lambda$, let $N_\lambda \leq n_\lambda$ be maximal such that $\bigcap_{n \leq N_\lambda} B^{(n)}_\lambda \neq \emptyset$. Pick $a_\lambda \in \bigcap_{n \leq N_\lambda} B^{(n)}_\lambda$. We claim that $(a_\lambda) \in B^{(N)}$ for every N. To see this, note that $\bigcap_{n \leq N} B^{(n)} = \lim_{\mathcal{U}} \bigcap_{n \leq N} B^{(n)}_\lambda$, so for most λ we have $\bigcap_{n \leq N} B^{(n)}_\lambda \neq \emptyset$. Also $n_\lambda \geq N$ for most λ. For these λ we have $N_\lambda \geq N$, so $a_\lambda \in \bigcap_{n \leq N_\lambda} B^{(n)}_\lambda \subseteq B^{(N)}_\lambda$. This shows that $(a_\lambda) \in B^{(N)}$ for all N, so $\bigcap_n B^{(n)} \neq \emptyset$.

(ii) For every $\lambda \in \Lambda$, let $N_\lambda \leq n_\lambda$ be maximal such that $\nu_\lambda(\bigcap_{n \leq N_\lambda} B^{(n)}_\lambda) > c$. Let $A_\lambda = \bigcap_{n \leq N_\lambda} B^{(n)}_\lambda$ and $A = \lim_{\mathcal{U}} A_\lambda$. Then we have $\nu(A) \geq c$, and we claim that $A \subseteq \bigcap_n B^{(n)}$. To see this, let N be a natural number and $(a_\lambda) \in A$. We have $c < \nu(\bigcap_{n \leq N} B^{(n)}) = \lim_{\mathcal{U}} \nu(\bigcap_{n \leq N} B^{(n)}_\lambda)$, so for most λ we have $\nu_\lambda(\bigcap_{n \leq N} B^{(n)}_\lambda) > c$. Also for most λ we have $n_\lambda \geq N$ and $a_\lambda \in A_\lambda$. For such λ we have $N_\lambda \geq N$. Then $a_\lambda \in A_\lambda = \bigcap_{n \leq N_\lambda} B^{(n)}_\lambda \subseteq B^{(N)}_\lambda$. So $(a_\lambda) \in B^{(N)}$, showing that $A \subseteq B^{(N)}$, and $A \subseteq \bigcap_n B^{(n)}$.

$\square$

Now we can construct a measure on $X_{\mathcal{U}}$.

Proposition 6.4.2. *Let $(\Lambda, \leq)$ be a directed set, and let $\mathcal{U}$ be a cofinal not σ-complete ultrafilter on Λ. For all λ, let X_λ be a set, let Σ_λ be an algebra on X_λ and let ν_λ be a finitely additive function on Σ_λ. Then there is a σ-additive measure ν, called the* Loeb measure*, on the σ-algebra $\sigma(\mathcal{A})$ generated by the quasiboxes in $X_{\mathcal{U}}$, such that*

$$\nu(\lim_{\mathcal{U}} B_\lambda) = \lim_{\mathcal{U}} \nu_\lambda(B_\lambda)$$

for every quasibox $\lim_{\mathcal{U}} B_\lambda$.

For every $A \in \sigma(\mathcal{A})$ with finite measure and $\varepsilon > 0$ there is a quasibox $B \subseteq A$ with $\nu(A \setminus B) < \varepsilon$.

Proof. We already have a finitely additive function on the algebra $\mathcal{A}$. It is trivially countably additive on $\mathcal{A}$: suppose we have a quasibox B that can be written as a disjoint union of nonempty quasiboxes $B^{(n)}$. Then the quasiboxes $X_{\mathcal{U}} \setminus B^{(n)}$ and B have the finite intersection property, so by Lemma 6.4.1(i), their intersection is empty, contradiction. So we can use Carathéodory's extension Theorem to extend ν to a σ-additive measure on $\sigma(\mathcal{A})$.

We say $A \in \sigma(\mathcal{A})$ is approachable by quasiboxes if for every $A' \in \sigma(\mathcal{A})$ with finite measure and for every $\varepsilon > 0$ there is a quasibox $B \subseteq A$ with finite measure and $\nu(A \setminus B \cap A') \leq \varepsilon$. We will show that all sets in $\sigma(\mathcal{A})$ are approachable by quasiboxes. Clearly all quasiboxes are approachable by quasiboxes, since we can take $B = A$. Since the quasiboxes are already closed under taking complements, we just have to show that this property is closed under countable intersections and countable unions.

So suppose $A = \bigcap_n A^{(n)}$ and the $A^{(n)}$ are approachable by quasiboxes. Let $A' \in \sigma(\mathcal{A})$ with finite measure. For all n let $B^{(n)} \subseteq A^{(n)}$ be a quasibox with finite measure satisfying $\nu(A^{(n)} \setminus B^{(n)} \cap A') \leq 2^{-n}\varepsilon$. By Lemma 6.4.1(ii) there is a quasibox $B \subseteq \cap_n B^{(n)}$ satisfying $\nu(B) \geq \nu(\cap_n B^{(n)}) - \varepsilon$. This quasibox satisfies $B \subseteq A$ and it necessarily has finite measure. Moreover we have

$$\nu(A \setminus B \cap A') \leq \varepsilon + \nu(A \setminus \cap_n B^{(n)} \cap A') \leq \varepsilon + \sum_n \nu(A^{(n)} \setminus B^{(n)} \cap A') \leq 2\varepsilon.$$

So A is approachable by quasiboxes.

Now let $A = \bigcup_n A^{(n)}$ and suppose that the $A^{(n)}$ are approachable by quasiboxes. Let $A' \in \sigma(\mathcal{A})$ with finite measure. There is N such that $\nu(\bigcup_{n \leq N} A^{(n)} \cap A') \geq \nu(A \cap A') - \varepsilon$. For $n \leq N$ let $B^{(n)} \subseteq A^{(n)}$ be a quasibox with finite measure satisfying $\nu(A^{(n)} \setminus B^{(n)} \cap A') \leq \frac{\varepsilon}{N}$. Let $B = \bigcup_{n \leq N} B^{(n)}$. This is a quasibox with finite measure and a subset of A. We have

$$\nu(A \setminus B \cap A') \leq \varepsilon + \nu(\bigcup_{n \leq N} A^{(n)} \setminus B \cap A') \leq \sum_{n \leq N} \nu(A^{(n)} \setminus B^{(n)} \cap A') + \varepsilon \leq 2\varepsilon.$$

This shows that A is approachable by quasiboxes.

Therefore, all $A \in \sigma(\mathcal{A})$ are approachable by quasiboxes. In particular, if $\nu(A)$ is finite we can take $A' = A$ and it follows that for every $\varepsilon > 0$ there is a quasibox $B \subseteq A$ with $\nu(A \setminus B) < \varepsilon$. □

Remark 6.4.3. In general, Carathéodory's extension Theorem does not say that there is a *unique* extension ν. However it gives a canonical way to construct this extension, and this is the one we will refer to as the Loeb measure.

6.5 Proof of Theorem 6.A

In this section we combine Proposition 6.3.2 with the results about the Loeb measure in the previous Section to prove Theorem 6.A. We restate the Theorem for convenience.

Theorem 6.A. Let G be a locally compact topological group. Then G is amenable if and only if for all controlled measure-preserving actions $G \curvearrowright (X, \nu)$ on a finitely-additive measure space there is an invariant mean on X.

Proof. Again the 'if' direction follows from Example 6.1.2. Let (X, ν) be a finitely-additive measure space and let $\alpha\colon G \curvearrowright X$ be a controlled measure-preserving action. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and let $\nu_{\mathcal{U}}$ be the Loeb measure on the ultraproduct $X_{\mathcal{U}}$. Then $(X_{\mathcal{U}}, \nu_{\mathcal{U}})$ is a σ-additive measure space. The group G acts on $X_{\mathcal{U}}$ and for any quasibox $B \subseteq X_{\mathcal{U}}$ and $g \in G$ it is easy to see that $\nu_{\mathcal{U}}(gB) = \nu_{\mathcal{U}}(B)$. From this it follows that the action $G \curvearrowright X_{\mathcal{U}}$ is measure-preserving.

Since α is controlled, the action on the ultraproduct is also controlled: let A be as in Definition 6.1.1. Then we consider the quasibox $\lim_{\mathcal{U}} A$, which has finite measure. For compact $K \subseteq G$ and $\varepsilon > 0$ there is a measurable $B \subseteq X$ with finite measure and $\nu(gA \cap B) \geq (1-\varepsilon)\nu(A)$ for $g \in G$. Then we take the quasibox $\lim_{\mathcal{U}} B$, this has the same measure as B and we see that $\nu_{\mathcal{U}}(g \lim_{\mathcal{U}} A \cap \lim_{\mathcal{U}} B) \geq (1-\varepsilon)\nu_{\mathcal{U}}(\lim_{\mathcal{U}} A)$.

By Proposition 6.3.2 there is an invariant mean on $X_{\mathcal{U}}$. Let $F \subseteq G$ be finite and $0 < \varepsilon < \frac{1}{2}$. By Proposition 6.3.1 there is a subset $A \subseteq X_{\mathcal{U}}$ such that $\nu_{\mathcal{U}}(gA \setminus A) < \varepsilon \nu_{\mathcal{U}}(A)$ for all $g \in F$. By Proposition 6.4.2(ii) there is a quasibox $B = \lim_{\mathcal{U}} B^{(n)}$ with $B \subseteq A$ and $\nu_{\mathcal{U}}(B) \geq (1-\varepsilon)\nu_{\mathcal{U}}(A)$. Then for $g \in F$ we have $\nu_{\mathcal{U}}(gB \setminus B) \leq \nu_{\mathcal{U}}(gA \setminus A) + \nu_{\mathcal{U}}(A \setminus B) < 2\varepsilon\nu_{\mathcal{U}}(A) \leq 4\varepsilon\nu_{\mathcal{U}}(B)$. Thus for most n we have $\nu(B^{(n)}) > 0$ and $\nu(gB^{(n)} \setminus B^{(n)}) < 4\varepsilon\nu(B^{(n)})$. Pick such an n and define the mean $\varphi_{F,\varepsilon}$ on X by $\varphi_{F,\varepsilon}(C) = \frac{\nu(C \cap B^{(n)})}{\nu(B^{(n)})}$. Then for $g \in F^{-1}$ we have $\varphi_{F,\varepsilon}(gC) = \frac{\nu(C \cap g^{-1}B^{(n)})}{\nu(B^{(n)})}$, and we get $|\varphi_{F,\varepsilon}(gC) - \varphi_{F,\varepsilon}(C)| \leq \frac{\nu(g^{-1}B^{(n)} \setminus B^{(n)})}{\nu(B^{(n)})} \leq 4\varepsilon$.

Taking a weak limit point of the $\varphi_{F,\varepsilon}$ as F increases to G and ε decreases to zero gives an invariant mean on X.

$\square$

7 Embeddings of sequences of graphs into Hilbert spaces

Chapter summary

We give a combinatorial obstruction for a sequence of graphs with bounded degree to coarsely embed into a Hilbert space. We introduce ρ-expanders and show that their covers do not coarsely embed into a Hilbert space. Finally, we discuss how it might be possible to use random graph methods to construct a sequence of graphs with bounded degree that cannot be embed into a Hilbert space, but can be embedded into an ℓ^p-space for all $p > 2$.

7.1 Introduction and main results

Let (X_n) be a sequence of graphs with uniformly bounded degree, such that the number of vertices of X_n converges to infinity. We will identify the graphs with their vertex sets and view them as a metric space, using the shortest path distance. Let X be the union of the X_n. Let (V, d) be a metric space. Recall that a map $F\colon X \to V$ a *coarse embedding* if there is a constant L and an increasing unbounded map $\rho\colon \mathbb{N} \to [0, \infty)$ such that for all n and $x, y \in X_n$ we have $d(F(x), F(y)) \leq L$ if x and y are adjacent, and $d(F(x), F(y)) \geq \rho(d(x, y))$, where $d(x, y)$ denotes the shortest path distance between x and y.

An interesting question is whether there exists a coarse embedding of X, when V can be chosen from a certain class of metric spaces. The article [50] addresses this question for Hilbert spaces. In fact, it gives a criterion for when it is possible to embed X coarsely into a Hilbert space. We will state it after giving the definition of *generalised expanders* from [50].

Definition 7.1.1 ([50]). Let (X_n) be a sequence of graphs with bounded degree and number of vertices going to infinity. The sequence is a *generalised expander* if there exists a symmetric measure α on $\bigsqcup_n X_n \times X_n$ satisfying the following conditions:

(i) We have $\alpha(X_n \times X_n) = 1$ for each n.

(ii) For all $R > 0$ we have $\alpha(E_R \cap X_n \times X_n) \to 0$ as $n \to \infty$, where $E_R = \{(x, y) \in X \mid d(x, y) \leq R\}$.

(iii) There is a constant K such that for all Hilbert spaces $\mathcal{H}$ and all L-Lipschitz functions $F\colon X \to \mathcal{H}$ we have the inequality

$$\sum_{x,y \in X_n} \alpha(x, y) \big\| F(x) - F(y) \big\|^2 \leq K L^2$$

for all n.

If a symmetric measure α satisfies the first two conditions, we call it a *far-away measure* for X.

Theorem 7.1.2 ([50], Theorem 1.3). *The following are equivalent:*

(i) The coarse space X does not embed coarsely into any Hilbert space.

(ii) There is a generalised expander (Y_n) and a coarse embedding $j\colon Y \to X$.

□

In fact, the criterion above is a bit more complicated than it needs to be: if $X = (X_n)$ allows a coarse embedding of a generalised expander, there is a subsequence of (X_n) that is itself a generalised expander (see Lemma 7.2.1).

Now we give a different algebraic and a combinatorial criterion that are obstructions for X to embed in a Hilbert space. If X is a box space they are equivalent criteria.

Definition 7.1.3. Let Γ be a residually finite group, finitely generated by a symmetric set S. Let $N_1, N_2, \ldots$ be a sequence of normal subgroups with increasing index and $\cap_n \Gamma_n = \{0\}$. Then each group Γ/Γ_n can be given a graph structure, where the class of g is adjacent to the class of $s \cdot g$, for $g \in \Gamma, s \in S$. The *box space* of Γ is the sequence of graphs (Γ/Γ_n). As a coarse space, it is independent of the choice of S (but not independent of the choice of the Γ_n).

We will use the Laplacian of the graph, and also the Laplacian of a measure on $\bigsqcup_n X_n \times X_n$.

Definition 7.1.4. Recall that the graph Laplacian is the operator Δ_X on $\ell^2 X$ given by $\Delta_X \xi(x) = \sum_{y \sim x}(\xi(x) - \xi(y))$, for $\xi \in \ell^2 X, x \in X$ and the sum is over the y that are adjacent to x. For a symmetric measure α on $\bigsqcup_n X_n \times X_n$, let Δ_α be the operator on $\ell^2 X$ given by

$$\Delta_\alpha \xi(x) = \sum_{y \in X_n} \alpha(x, y)(\xi(x) - \xi(y))$$

for $\xi \in \ell^2 X, x \in X_n$. Note that Δ_X and Δ_α are always positive operators.

Proposition 7.1.5. *Let $X = (X_n)$ be a sequence of graphs with bounded degree and increasing number of edges. Consider the following conditions on X.*

(i) There is no coarse embedding of X into a Hilbert space.

(ii) Some subsequence of (X_n) is a generalised expander.

(iii) There is a far-away measure α for a subsequence of (X_n), and a constant c, such that the inequality $|X_n| \cdot \Delta_\alpha \leq c \cdot \Delta_X$ holds in $B(\ell^2 X_n)$ for all n.

(iv) There is a far-away measure α for a subsequence of (X_n), and a constant c, such that for all $A \subseteq X_n$ we have

$$|X_n| \cdot \alpha(A \times X_n \setminus A) \leq c \cdot E_X(A, X_n \setminus A).$$

Here $E_X(A, B)$ denotes the number of edges between subsets A, B of X.

We have (iii) $\implies$ (iv) $\implies$ (i) $\implies$ (ii). Moreover, for box spaces, the implication (ii) $\implies$ (iii) holds, meaning that all the conditions are equivalent.

We will prove this proposition in the next section. Now we will introduce ρ-expanders. Recall from Chapter 3 that $Z(X)$ denotes the cycle space of a graph X. Let m be a positive integer and let $\rho\colon Z(X) \to (\mathbb{Z}/2)^m$ be a group morphism. Then we can construct the *ρ-cover of X*, denoted by $\tilde{X}$, defined as follows: let E be the set of edges of X and let T be the set of edges of a spanning tree. Then we can identify $Z(X)$ with the free abelian group generated by $E \setminus T$. The vertex set of $\tilde{X}$ is $X \times (\mathbb{Z}/2)^m$. Two vertices $(x, r), (y, r)$ are connected if $(x, y) \in T$, and $(x, r), (y, s)$ are connected if $(x, y) \in E \setminus T$ and $s = r + \rho((x, y))$.

Let $\pi\colon \tilde{X} \to X$ be the projection sending (x, r) to x. It can be shown that the graph $\tilde{X}$ does not depend on the choice of the spanning tree T, up to isomorphism preserving π (see [26]). For $\tilde{x} = (x, r) \in \tilde{X}$, and $s \in (\mathbb{Z}/2)^m$, let $s \cdot \tilde{x} = (x, s + r)$.

Now let (X_n) be a sequence of graphs and $\rho_n\colon Z(X) \to (\mathbb{Z}/2)^{m_n}$. We are interested whether the sequence $\tilde{X}_n$ embeds coarsely into a Hilbert space. In [26] it is shown that this is the case when ρ_n is the projection $Z(X) \to (\mathbb{Z}/2)^{r_n}$ where r_n is the rank of $Z(X)$. We now give a combinatorial obstruction that prevents coarse embedding into a Hilbert space. First we give some preliminary definitions.

Definition 7.1.6. Let X be a graph and $\rho\colon Z(X) \to (\mathbb{Z}/2)^m$ a map. The ρ-girth of X is the length of the minimal loop γ with $\rho(\gamma) \neq 0$.

Definition 7.1.7. For $\chi \in ((\mathbb{Z}/2)^m)^*$, let $\tau(\chi) = \#\{i \in \{1, \ldots, m\} \colon \chi(e_i) = -1\}$ where e_i denotes the i-th basis vector of $(\mathbb{Z}/2)^m$.

Definition 7.1.8. Let A be a (not necessarily induced) subgraph of X. We denote by $|A|$ the number of vertices in A and by ∂A the number of edges not in A, but with at least one vertex in A. We will view $Z(A)$ as a subgroup of $Z(X)$.

Definition 7.1.9. Let (X_n) be a sequence of graphs with uniformly bounded degrees, with maps $\rho_n\colon Z(X_n) \to (\mathbb{Z}/2)^{m_n}$. We call it a ρ-expander if the following conditions are satisfied:

(i) The ρ_n-girth of X_n tends to infinity.

(ii) There is a constant $c > 0$, such that for every n, every $\chi \in ((\mathbb{Z}/2)^{m_n})^*$, and every subgraph A of X_n satisfying $\chi \circ \rho_{n|Z(A)} = 1$, the inequality

$$\partial A \geq c\sqrt{\frac{\tau(\chi)}{m_n}}|A|$$

holds.

Now we can show that the ρ-cover of a ρ-expander does not coarsely embed into a Hilbert space. Let α_n be the measure on $\tilde{X}_n \times \tilde{X}_n$ given by

$$\alpha_n(\tilde{x}, e_i \cdot \tilde{x}) = \frac{1}{m|\tilde{X}_n|}$$

for $1 \leq i \leq m$, and $\alpha(\tilde{x}, \tilde{y}) = 0$ for all other pairs. If the ρ_n-girth of X_n tends to infinity, then the union α of the α_n is a far-away measure.

Theorem 7.A. Let (X_n) be a sequence of graphs with uniformly bounded degrees, with maps $\rho_n\colon Z(X) \to (\mathbb{Z}/2)^{m_n}$. If (X_n) is a ρ-expander then the sequence $\tilde{X}_n$ is a generalised expander with the far-away measure α. In particular, it does not coarsely embed into a Hilbert space.

We do not have interesting examples of ρ-expanders. Possibly they could be constructed using probabilistic methods, similar to the way the first examples of expander sequences were constructed. For this we would need a model of a random graph embedded in a torus $\mathbb{T}^m$. This would then give rise to a map $Z(X) \to H_1(\mathbb{T}^m) = \mathbb{Z}^m \to (\mathbb{Z}/2)^m$. We introduce a model of random graphs embedded in an arbitrary manifold in the next chapter. However, we can only describe these random graphs aptly when the manifold is $\mathbb{R}^m$.

7.2 Obstructions for coarse embeddability into a Hilbert space

In this Section we prove Proposition 7.1.5. First we show that any sequence of graphs that allows an embedding of a generalised expander is already a generalised expander, up to taking a subsequence.

Lemma 7.2.1. *Let $X = (X_n)$ and $Y = (Y_n)$ be sequences of graphs, with bounded degree and increasing number of vertices. Suppose that $j\colon Y \to X$ is a coarse embedding sending Y_n to X_n, and suppose that Y is a generalised expander. Then X is also a generalised expander.*

Proof. Let β be a measure on $\bigsqcup_n Y_n \times Y_n$ satisfying the conditions from Definition 7.1.1, for some constant K. Define the measure α on $\bigsqcup_n X_n \times X_n$ by $\alpha(A) = \beta((j\times j)^{-1}(A))$ for $A \subseteq \bigsqcup_n X_n \times X_n$. We check now that this measure satisfies the conditions from Definition 7.1.1. We see directly that $\alpha(X_n \times X_n) = \beta(Y_n \times Y_n) = 1$ for all n. For any controlled set $E \subseteq X \times X$ and any n we have $\alpha(E \cap X_n \times X_n) = \beta((j \times j)^{-1}(E) \cap Y_n \times Y_n)$. Since j is a coarse embedding, the set $(j\times j)^{-1}(E) \subseteq Y \times Y$ is a controlled set, so $\beta((j\times j)^{-1}(E) \cap Y_n \times Y_n)$ converges to 0 as $n \to \infty$.

Lastly, since j is a coarse embedding, it is M-Lipschitz for some M. Let $F\colon X \to \mathcal{H}$ be an L-Lipschitz map into some Hilbert space. Then $F \circ j$ is LM-Lipschitz, and we get

$$\sum_{x,x'\in X} \alpha(x,x')\big\|F(x) - F(x')\big\|^2 = \sum_{y,y'\in Y} \beta(y,y')\big\|Fj(y) - Fj(y')\big\|^2 \leq KM^2L^2.$$

So α satisfies the conditions from Definition 7.1.1 with the constant KM^2, hence X is a generalised expander. ☐

Proof of Proposition 7.1.5. The equivalence (i) $\iff$ (ii) follows from Theorem 7.1.2 and Lemma 7.2.1.

Suppose we have (iii). Then for each $\xi \in \ell^2 X_n$ we have $\langle \Delta_\alpha \xi, \xi\rangle \leq c\langle \Delta_X \xi, \xi\rangle$. If we apply this on an indicator function $\xi = \mathbb{1}_A$ with $A \subseteq X_n$, we get $|X_n| \cdot \alpha(A \times X_n \setminus A) \leq c \cdot E_X(A, X_n \setminus A)$. Hence (iv) is true.

Suppose we have (iv), and that there is a coarse embedding $F\colon X \to \mathcal{H}$ for some Hilbert space $\mathcal{H}$. Let n be a positive integer. Since $F(X_n) \subseteq \mathcal{H}$ is a finite set, it is isometric to a subset of $\mathbb{R}^m$ for some finite m. Therefore we assume without loss of generality that $F(X_n) \subseteq \mathbb{R}^m$. For any unit vector $v \in \mathbb{R}^m$ and $t \in \mathbb{R}$ let $A_{v,t} = \{x \in X_n \mid F(x)\cdot v \leq t\}$. We then know that $|X_n|\alpha(A_{v,t} \times X_n \setminus A_{v,t}) \leq c \cdot E_X(A_{v,t}, X_n \setminus A_{v,t})$. Integrating this inequality from $-\infty$ to ∞ gives

$$\begin{aligned}
|X_n| \sum_{x,y} \alpha(x,y)|F(x)\cdot v - F(y)\cdot v| &= 2|X_n| \int_{-\infty}^{\infty} \sum_{F(x)\cdot v \leq t, F(y)\cdot v \geq t} \alpha(x,y)dt \\
&= 2|X_n| \int_{-\infty}^{\infty} \alpha(A_{v,t} \times X_n \setminus A_{v,t})dt \\
&\leq 2c \int_{-\infty}^{\infty} E_X(A_{v,t}, X_n \setminus A_{v,t})dt \\
&= 2c \int_{-\infty}^{\infty} \sum_{x\sim y, F(x)\cdot v\leq t, F(y)\cdot v \geq t} dt \\
&= c \sum_{x\sim y} |F(x)\cdot v - F(y)\cdot v|.
\end{aligned}$$

Now we will choose v uniformly at random from the unit sphere. There is some constant b, depending on m, such that $\mathbb{E}_v|w\cdot v| = b\|w\|$ for all $w \in \mathbb{R}^m$. Now

$$|X_n| \sum_{x,y} \alpha(x,y)\big\|F(x) - F(y)\big\| = \frac{|X_n|}{b} \sum_{x,y} \mathbb{E}_v \alpha(x,y)|F(x)\cdot v - F(y)\cdot v|$$

$$
\begin{aligned}
&\le \frac{c}{b}\mathbb{E}_v \sum_{x\sim y} |F(x)\cdot v - F(y)\cdot v| \\
&= c\sum_{x\sim y} \left\|F(x)-F(y)\right\|.
\end{aligned}
$$

However, since F is a coarse embedding, we have $\left\|F(x)-F(y)\right\| \le L$ for some constant L. So this gives $\sum_{x,y} \alpha(x,y)\left\|F(x)-F(y)\right\| \le \frac{1}{2}cdL$, where d is the maximal degree of any vertex in X. There is some $R > 0$ such that for all x, y with $d(x,y) \ge R$ we have $\left\|F(x)-F(y)\right\| \ge cdL$. Now if we pick n large enough that $\alpha(E_R \cap X_n \times X_n) < \frac{1}{3}$, we have $\sum_{x,y} \alpha(x,y)\left\|F(x)-F(y)\right\| \ge \frac{2}{3}cdL$, and we have a contradiction. This shows (iv) $\Longrightarrow$ (i).

Finally, suppose that X is the box space of a group Γ, with the sequence of subgroups Γ_n, so $X_n = \Gamma/\Gamma_n$. Suppose moreover that (ii) holds for some measure α and constant K. We can assume that α is invariant for the left-action of Γ on X, by replacing it with its average. Let $v \in \ell^2 X_n$ and define the map

$$
F\colon X_n \to \mathbb{R}^{X_n},\ F(x)(z) = v(z\cdot x)
$$

where $x, z \in X_n$. Then we have for all $x \in X_n, s \in S$ the inequality

$$
\left\|F(x)-F(x\cdot s)\right\|^2 = \sum_{z\in X_n} |v(z\cdot x)-v(z\cdot x\cdot s)|^2 = \sum_{y\in X_n} |v(y)-v(y\cdot s)|^2 \le \langle \Delta_\Gamma v, v\rangle,
$$

so F is Lipschitz with the constant $\langle \Delta_\Gamma v, v\rangle^{\frac{1}{2}}$. On the other hand, we have

$$
\begin{aligned}
\sum_{x,y\in X_n} \alpha(x,y)\left\|F(x)-F(y)\right\|^2 &= \sum_{x,y,z\in X_n} \alpha(x,y)|v(z\cdot x)-v(z\cdot y)|^2 \\
&= |X_n| \sum_{x,y\in X_n} \alpha(x,y)|v(x)-v(y)|^2 \\
&= |X_n|\langle \Delta_\alpha v, v\rangle.
\end{aligned}
$$

So by assumption we know that $|X_n|\langle \Delta_\alpha v, v\rangle \le K\langle \Delta_\Gamma v, v\rangle$. This shows (iii) with the constant $c = K$. ☐

Remark 7.2.2. If we take α to be the uniform probability measure on each $X_n \times X_n$, then conditions (ii),(iii),(iv) are well-known different criteria to be an expander.

Remark 7.2.3. For (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (ii) we can take the same measure each time. However, even if (iii) holds, there may be far-away measures α for which (iv) holds and (iii) does not hold (see Example 7.2.4).

Example 7.2.4. Let $Y = (Y_n)$ be an expander sequence, with maximal degree d and $E_Y(A, Y_n \setminus A) \ge \frac{1}{c|Y_n|}|A|(|Y_n| - |A|)$ for $A \subseteq Y_n$. Let $m = |Y_n|$ and label two different points in Y_n by y_0, y_1.

Construct the graph X_n as follows: take n disjoint copies of Y_n, labelled $(Y_n)_1, \dots, (Y_n)_n$. Let $(y_i)_k$ be the point y_i in $(Y_n)_k$, for $i \in \{0,1\}$ and $1 \le k \le n$. Now, in addition to the edges inside of the $(Y_n)_k$, connect the point $(y_1)_k$ with the point $(y_0)_{k+1}$, for $1 \le k \le n-1$. If we do this for all n, we get a sequence $X = (X_n)$ of graphs with degree at most $d+1$, and increasing number of vertices. We will now define a far-away measure on X that satisfies condition (iv) but not condition (iii) above.

Define the measure α on $X_n \times X_n$ as follows: for $x, y \in (Y_n)_k$, set $\alpha(x,y) = \frac{mn-1}{m^3n^2}$. Moreover, set $\alpha((y_0)_1,(y_1)_n) = \frac{1}{mn}$, and set $\alpha(x,y) = 0$ for all other pairs. Note that α is a far-away measure.

Let $A \subseteq X_n$ be a non-empty proper subset. Since X_n is connected, we have $E_X(A, X_n \setminus A) \geq 1$. Write $A = A_1 \cup \cdots \cup A_n$, with $A_k = A \cap (Y_n)_k$. Then

$$cE_X(A, X_n \setminus A) \geq c \sum_{k=1}^{n} E_X(A_k, (Y_n)_k \setminus A_k) \geq \sum_{k=1}^{n} \frac{1}{m}|A_k|(m - |A_k|)$$

while

$$|X_n|\alpha(A \times X_n \setminus A) \leq 1 + \sum_{k=1}^{n} \frac{1}{m}|A_k|(m - |A_k|).$$

Hence $|X_n|\alpha(A \times X_n \setminus A) \leq (1+c)E_X(A, X_n \setminus A)$. So condition (iv) of Proposition 7.1.5 holds for α.

However, consider the function $\xi \in \ell^2 X_n$ given by $\xi(x) = k$ for $x \in (Y_n)_k$. Then $|X_n|\langle \Delta\alpha\xi, \xi\rangle = n^2$ while $\langle \Delta_X \xi, \xi \rangle = n-1$. So condition (iii) does not hold for α.

7.3 Proof of Theorem 7.A

Now we can prove Theorem 7.A, which we restate for convenience.

Theorem 7.A. Let (X_n) be a sequence of graphs with uniformly bounded degrees, with maps $\rho_n \colon Z(X) \to (\mathbb{Z}/2)^{m_n}$. If (X_n) is a ρ-expander then the sequence $\tilde{X}_n$ is a generalised expander with the far-away measure α. In particular, it does not coarsely embed into a Hilbert space.

Proof. Let c be the constant of Definition 7.1.9. We need to show that $c'\Delta_{\tilde{X}_n} \geq |X_n|\Delta_\alpha$. These are operators on $L^2\tilde{X}_n$. We have a decomposition $L^2\tilde{X}_n = \bigoplus_\chi L^2_\chi \tilde{X}_n$ where the sum ranges over all characters $\chi \colon (\mathbb{Z}/2\mathbb{Z})^m \to \mathbb{Z}/2$, and $L^2_\chi\tilde{X}_n$ consists of all functions $f \colon \tilde{X}_n \to \mathbb{C}$ satisfying $f(s \cdot \tilde{x}) = \chi(s)f(\tilde{x})$ for $\tilde{x} \in \tilde{X}_n$ and $s \in (\mathbb{Z}/2\mathbb{Z})^m$. For $f \in L^2_\chi\tilde{X}_n$ we have $\Delta_\alpha f(\tilde{x}) = \frac{1}{m|\tilde{X}_n|}\sum_i f(e_i \cdot \tilde{x}) = \frac{1}{m|\tilde{X}_n|}\sum_i \chi(e_i)f(\tilde{x}) = \frac{\tau(\chi)}{m|\tilde{X}_n|}f(\tilde{x})$. So on $L^2_\chi\tilde{X}_n$ we have $|\tilde{X}_n|\Delta_\alpha = \frac{\tau(\chi)}{m}$. We need to show that for $f \in L^2_\chi\tilde{X}_n$ we have $\langle \Delta_{\tilde{X}_n} f, f\rangle \geq c'\frac{\tau(\chi)}{m}\|f\|^2$.

So let $f \in L^2_\chi(\tilde{X}_n)$ and consider the set of "forbidden edges"

$$\tilde{F} = \{(\tilde{x}, \tilde{y}) \in \tilde{E} \colon f(\tilde{x}) \text{ and } f(\tilde{y}) \text{ have different signs, or one of them is } 0\},$$

where $\tilde{E}$ denotes the set of edges in $\tilde{X}$. Let $F = \pi_*(\tilde{F})$. Let $\gamma = \tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_k$ be a path in $\tilde{X}$ with $\pi(\tilde{x}_k) = \pi(\tilde{x}_0)$. Then $\tilde{x}_k = \rho(\pi_*\gamma) \cdot \tilde{x}_0$. So if $\chi(\rho(\pi_*\gamma) = -1$ then $f(\tilde{x}_k) = -f(\tilde{x}_0)$. In this case, γ has to contain a forbidden edge. So any subgraph A of X that does not contain an edge in F satisfies the condition $\chi \circ \rho_{|Z(A)} = 1$.

For $t > 0$ let A_t be the graph with vertex set $\{x \in X \colon f(x,0)^2 \geq t\}$ and all the edges between these vertices, except the ones in F. Then we have

$$\partial A_t \geq c\sqrt{\frac{\tau(\chi)}{m}}|A_t|. \tag{7}$$

We can estimate the left-hand side as

$$\partial A_t = \#\{x \sim y \colon f(x,0)^2 \leq t \leq f(y,0)^2\} + \#\{(x,y) \in F \colon f(x,0)^2 \geq t, f(y,0)^2 \geq t\}$$

$$
\begin{aligned}
&= 2^{-m}\#\{\tilde{x}\sim\tilde{y}\colon f(\tilde{x})^2 \le t \le f(\tilde{y})^2\} + 2^{-m}\#\{(\tilde{x},\tilde{y})\in\tilde{F}\colon f(\tilde{x})^2\ge t, f(\tilde{y})^2\ge t\}\\
&\le 2^{-m}\#\{\tilde{x}\sim\tilde{y}\colon f(\tilde{x})^2 \le t \le f(\tilde{y})^2\} + \#\{x\sim y\colon |f(\tilde{x})-f(\tilde{y})|\ge 2t^{\frac{1}{2}}\}.
\end{aligned}
$$

We have

$$\int_0^\infty \#\{\tilde{x}\sim\tilde{y}\colon f(\tilde{x})^2\le t\le f(\tilde{y})^2\}dt = \sum_{\tilde{x}\sim\tilde{y}} |f(\tilde{x})^2 - f(\tilde{y})^2|$$

and

$$\int_0^\infty \#\{\tilde{x}\sim\tilde{y}\colon |f(\tilde{x})-f(\tilde{y})|\ge 2t^{\frac{1}{2}}\}dt = \frac{1}{4}\sum_{\tilde{x}\sim\tilde{y}}|f(\tilde{x})-f(\tilde{y})|^2$$

and

$$\int_0^\infty |A_t|dt = \sum_x f(x,0)^2 = 2^{-m}\sum_{\tilde{x}} f(\tilde{x})^2.$$

Combining these equalities and integrating inequality (7) from 0 to ∞ gives

$$\sqrt{\frac{\tau(\chi)}{m}}\sum_{\tilde{x}\in\tilde{X}} f(\tilde{x})^2 \le \frac{1}{c}\sum_{\tilde{x}\sim\tilde{y}}|f(\tilde{x})^2-f(\tilde{y})^2| + \frac{1}{4c}\sum_{\tilde{x}\sim\tilde{y}}|f(\tilde{x})-f(\tilde{y})|^2. \tag{8}$$

We will use the Cauchy-Schwartz inequality to estimate the first term on the right-hand side:

$$
\begin{aligned}
\sum_{\tilde{x}\sim\tilde{y}}|f(\tilde{x})^2-f(\tilde{y})^2| &= \sum_{\tilde{x}\sim\tilde{y}}|f(\tilde{x})-f(\tilde{y})|\cdot|f(\tilde{x})+f(\tilde{y})|\\
&\le \left(\sum_{\tilde{x}\sim\tilde{y}}|f(\tilde{x})-f(\tilde{y})|^2\right)^{\frac{1}{2}}\left(\sum_{\tilde{x}\sim\tilde{y}}|f(\tilde{x})+f(\tilde{y})|^2\right)^{\frac{1}{2}}\\
&\le \sqrt{2}\left(\sum_{\tilde{x}\sim\tilde{y}}|f(\tilde{x})-f(\tilde{y})|^2\right)^{\frac{1}{2}}\left(\sum_{\tilde{x}\sim\tilde{y}}f(\tilde{x})^2+f(\tilde{y})^2\right)^{\frac{1}{2}}\\
&\le \sqrt{2d}\left(\sum_{\tilde{x}\sim\tilde{y}}|f(\tilde{x})-f(\tilde{y})|^2\right)^{\frac{1}{2}}\left(\sum_{\tilde{x}}f(\tilde{x})^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Combining this inequality with inequality (8) we get

$$\sqrt{\frac{\tau(\chi)}{m}}\sum_{\tilde{x}} f(\tilde{x})^2 \le \frac{\sqrt{2d}}{c}\left(\sum_{\tilde{x}\sim\tilde{y}}|f(\tilde{x})-f(\tilde{y})|^2\right)^{\frac{1}{2}}\left(\sum_{\tilde{x}}f(\tilde{x})^2\right)^{\frac{1}{2}} + \frac{1}{4c}\sum_{\tilde{x}\sim\tilde{y}}|f(\tilde{x})-f(\tilde{y})|^2.$$

Now note that $\|f\|^2 = \sum_{\tilde{x}} f(\tilde{x})^2$ while $\langle\Delta_{\tilde{X}_n} f, f\rangle = \sum_{\tilde{x}\sim\tilde{y}}|f(\tilde{x})-f(\tilde{y})|^2$. Then the inequality above says that

$$\sqrt{\frac{\tau(\chi)}{m}}\|f\|^2 \le \frac{\sqrt{2d}}{c}\|f\|\left(\langle\Delta_{\tilde{X}_n} f, f\rangle\right)^{\frac{1}{2}} + \frac{1}{4c}\langle\Delta_{\tilde{X}_n} f, f\rangle.$$

Some straight-forward computation gives $c' > 0$ only depending on c and d such that

$$\langle\Delta_{\tilde{X}_n} f, f\rangle \ge c'\frac{\tau(\chi)}{m}\|f\|^2.$$

So we have shown that $\tilde{X}_n$ is a generalised expander with far-away measure α. By Proposition 7.1.5, we know that it does not embed into any Hilbert space. $\square$

7.4 Embeddings in ℓ^p

A fundamental open question in the field (see e.g. [2]) is: does there exist a sequence of graphs with bounded degree that cannot be coarsely embedded into a Hilbert space, but can be embedded into an ℓ^p-space for some $p > 2$? In this section we briefly sketch a possible approach to answer this question. This approach would use the probabilistic method, and in particular it hinges on a model of a random graph together with an embedding into a high-dimensional torus. No applicable models have been studied in the literature. We make an attempt to develop one in the next chapter of this thesis. It turns out to be very difficult to analyse the small-scale and large-scale behaviour of the random graphs produced by this model. Therefore, the problem remains open.

We would like to create a sequence of graphs X_m, on $n(m)$ vertices, together with a function $\rho_m\colon Z(X_m) \to (\mathbb{Z}/2\mathbb{Z})^m$, such that the cover $(\tilde{X}_m)$ does not coarsely embed into a Hilbert space but does allow a coarse embedding into an ℓ^p-space for all $p > 2$. To explain the approach, we assume that we have a model of a "random d-regular graph X_m on $n(m)$ vertices together with a function $f\colon X_m \to \mathbb{T}^m$". We will let $n(m)$ grow exponentially fast in m. Moreover we will assume that this model satisfies some desirable properties. In particular, for all $x \in X$ the point $f(x)$ should be uniformly distributed in $\mathbb{T}^m$. For an edge (x, y) we denote by $v_{x,y} \in \mathbb{R}^m$ the lift of $f(y) - f(x)$ with smallest possible length. This should be (approximately) jointly normally distributed in $\mathbb{R}^m$ with a mean of 0 and a mean length of 1. From this we can already conclude that if we have a sequence X_m of random d-regular graphs with a function to $\mathbb{T}^m$, that with probability one, the embedding $f\colon X_m \to \mathbb{T}^m$ will be L-Lipschitz for some L. Thus the map f will be controlled. Now, we should assume additionally that f is a coarse embedding, i.e. there is a proper non-decreasing function $\rho\colon \mathbb{N} \to \mathbb{R}$ such that for all m and $x, y \in X_m$ we have $\|f(x) - f(y)\| \geq \rho(d(x, y))$.

The embedding f will give rise to a function ρ as follows: the $v_{x,y} \in \mathbb{R}^m$ give a function $\mathbb{R}[E(X)] \to \mathbb{R}^m$. This function takes integer values on cycles. So after composing with the projection $\mathbb{Z}^m \to (\mathbb{Z}/2\mathbb{Z})^m$ we get a map $\rho\colon \mathbb{Z}(X) \to (\mathbb{Z}/2\mathbb{Z})^m$. Now the most demanding assumption we will make of our random model is that it produces ρ-expanders with high probability. This ensures that the ρ-cover $(\tilde{X}_m)$ cannot be embedded into a Hilbert space.

Now we explain how, for any $p > 2$, the ρ-cover $(\tilde{X}_m)$ can be embedded into an ℓ^p-space. For this we note that the ℓ^p-norm of a normally distributed variable in m dimensions is approximately $c_m = m^{\frac{1}{p}-\frac{1}{2}}$. In fact, a careful analysis shows that this norm is with high probability within a constant factor of c_m. Now consider the torus $(\mathbb{R}/2\mathbb{Z})^m$ (note the extra factor 2). We give this the metric coming from the ℓ^p-norm, scaled by c_m. We can imagine X_m sitting inside the image of $[0, 1]^m$ inside this torus. Now we can embed $\tilde{X}_m$ into this torus by sending (x, s) to $x + s$. Assuming a suitable model of X_m, this will be a coarse embedding with high probability. Finally the torus can be coarsely embedded into an ℓ^p-space of dimension $2m$.

8 Random graph embeddings

Chapter summary

We introduce a model for random regular graphs G together with a function from the vertex set to a manifold M. We study the asymptotic behaviour of random graphs embedded in $\mathbb{R}^m$ as $m \to \infty$, and their local asymptotic behaviour in particular.

8.1 Introduction and main results

Let $d \geq 3$ be a fixed integer. Let M be either a compact Riemannian manifold or a Riemannian manifold with a basepoint. In this chapter we will consider a novel way to construct random d-regular graphs embedded in the manifold M. We will then specialise to the case $M = \mathbb{R}^m$, which is the only non-trivial case where we have any idea what happens. We will consider d-regular graphs on n vertices, and we will always assume that dn is even.

Let m be the dimension of M and let d be the metric on M. In this chapter we will consider varying graphs on a fixed vertex sets $X = \{1, 2, \ldots, n\}$. The graphs will be denoted by $G = (X, E)$ where E is the set of edges. For integers n we consider the space $\mathrm{GraphFunctions}(n, M)$, whose elements consist of a connected d-regular graph $G = (X, E)$ on $X = \{1, 2, \ldots, n\}$ together with a function $\gamma \colon X \to M$. If M has a basepoint we also require that $\gamma(1)$ is that basepoint. The space $\mathrm{GraphFunctions}(n, M)$ is a manifold, isomorphic to a union of copies of M^n (if there is no base-point) or M^{n-1} (if there is a base-point). For any element $(G, \gamma) \in \mathrm{GraphFunctions}(n, M)$ we define the *energy*

$$E(G, \gamma) = \sum_{(x,y) \in E} d(\gamma(x), \gamma(y))^2.$$

Then we define the *intensity* of this embedding to be

$$\varphi_0(G, \gamma) = \exp(-mE(G, \gamma)).$$

For a large class of manifolds the integral $\int_{(G,\gamma) \in \mathrm{GraphFunctions}(n,M)} \varphi_0(G, \gamma)$ converges, so we can define

$$\varphi(G, \gamma) = \frac{\varphi_0(G, \gamma)}{\int_{(G,\gamma) \in \mathrm{GraphFunctions}(n,M)} \varphi_0(G, \gamma)}.$$

Define also $\varphi_0(G) = \int_\gamma \varphi_0(G, \gamma)$ and $\varphi(G) = \int_\gamma \varphi(G, \gamma)$.

Now φ defines a probability distribution on $\mathrm{GraphFunctions}(n, M)$. We shall call a random variable in $\mathrm{GraphFunctions}(n, M)$ chosen according to this probability distribution a *random graph embedded in M*. From now on we will always take $M = \mathbb{R}^m$ for some m. We will consider the asymptotic behaviour of an sequence G_m of d-regular graphs on $n(m)$ vertices independently randomly embedded in $\mathbb{R}^m$, as $m \to \infty$. It turns out that the local asymptotic behaviour of such a sequence depends on the growth rate of n as a function of m.

8.1.1 Embeddings in $\mathbb{R}^m$

Consider the manifold $M = \mathbb{R}^m$ with base-point zero. For a d-regular graph G on n vertices, let $T(G)$ be the number of spanning trees of G. Then we have $\varphi(G, \mathbb{R}^m) \sim T(G)^{-\frac{m}{2}}$, i.e. there is a constant C, which may depend on d, m and n but not on the graph G, such that $\varphi(G, \mathbb{R}^m) = CT(G)^{\frac{m}{2}}$ (see Proposition 8.2.1).

The number of spanning trees is not hard to determine algorithmically. However, to generate a random graph embedded in $\mathbb{R}^m$ in practice, one needs to list all d-regular graphs first and compute their number of spanning trees, which is infeasible. We do not know of a practical way to generate a random graph embedded in $\mathbb{R}^m$. However, once the graph G is known, it is not difficult to generate a random embedding of it in $\mathbb{R}^m$ (see Proposition 8.2.3).

8.1.2 Local ensembles

Let (G_m) be a sequence of graphs. To study the local behaviour of (G_m) we introduce the following notation. We use the shortest path metric on the vertices of the graphs. Also, for a vertex x and an edge e, the distance between x and e is the shortest distance between x and one of the endpoints of e. We denote by $B_R(x)$ (resp. $B_R(e)$) the induced subgraph on all vertices of distance at most R from x (resp. e).

Definition 8.1.1. For an integer $R \geq 1$, let $\mathcal{L}_R$ be the set of isomorphism classes of graphs A with a designated edge e_* satisfying the following conditions:

(i) All vertices of A are at distance at most R from e_*.

(ii) The degree of all vertices is at most d.

(iii) For all vertices of A with a distance smaller than R from e_* the degree is exactly d.

(iv) There is a vertex with degree smaller than d.

These are exactly the graphs that can occur as an R-ball around an edge in a d-regular graph with diameter larger than $2R$. Let $\mathcal{P}(\mathcal{L}_R)$ denote the set of probability distribution on $\mathcal{L}_R$.

Definition 8.1.2. Let G be a finite d-regular graph with a diameter larger than $2R$. For any edge e of G we have $(B_R(e), e) \in \mathcal{L}_R$. Picking e uniformly at random gives a probability distribution $\rho_R(G) \in \mathcal{P}(\mathcal{L}_R)$.

For $R' > R$ we have a restriction map from $\mathcal{L}_{R'}$ to $\mathcal{L}_R$. Let $\mathcal{L}$ be the set of all isomorphism classes of infinite connected d-regular graphs with a designated edge. This is the direct limit of the $\mathcal{L}_R$. Let $\mathcal{P}(\mathcal{L})$ be the direct limit of the $\mathcal{P}(\mathcal{L}_R)$, which is also a set of probability distributions on $\mathcal{L}$. An element of $\mathcal{P}(\mathcal{L})$ is called a *local ensemble*. For $\mu \in \mathcal{P}(\mathcal{L})$ we denote by μ_R the projection to $\mathcal{P}(\mathcal{L}_R)$.

For μ_R, μ'_R and $\varepsilon > 0$ we write $\mu_R \overset{\varepsilon}{\sim} \mu'_R$ if $|\mu_R(S) - \mu'_R(S)| < \varepsilon$ for all subsets $S \subseteq \mathcal{L}_R$. The following notion of local convergence was introduced in [4].

Definition 8.1.3. Let (G_m) be a sequence of d-regular graphs with number of vertices going to infinity. Let μ be a local ensemble. We say that (G_m) *converges locally* to μ if for all $\varepsilon > 0$ and $R > 0$ there is an N such that for $n \geq N$ the diameter of G_m is at least $2R$ and $\rho_R(G_m) \overset{\varepsilon}{\sim} \mu_R$.

Not every probability distribution in $\mathcal{P}(\mathcal{L})$ is the local limit of some sequence of finite d-regular graphs; those that are, are called *sofic ensembles*. We give some examples of sofic ensembles.

Examples 8.1.4. (i) There is a unique element $(T, e_*) \in \mathcal{L}_R$ where T is a tree. The *free ensemble* μ_{free} is the limit of the delta distributions on $\mathcal{P}(\mathcal{L}_R)$ supported on these trees. It is the local limit of any sequence of d-regular graphs with large girth.

(ii) Let H be a finite graph where every vertex has degree d except for two vertices x_0, x_1 which have degree $d-1$. Consider the graph G which consists of countably many copies of H indexed by $\mathbb{Z}$, where moreover the x_1 in H_n is connected to the x_0 in H_{n+1}. This is a d-regular graph. We get a probability measure on $\mathcal{L}_R$ by taking a uniformly at random one of the edges of H_0 or the edge between H_0 and H_1, and considering its R-neighbourhood. The limit as $R \to \infty$ is the *linear ensemble* λ_H. It is a sofic ensemble: let G_k be the quotient graph we get by identifying H_n with H_{n+k} for all n. Then the G_k converge locally to λ_H. This construction also works if all vertices of H have degree d except one which has degree $d-2$.

We will try to describe the local behaviour of random graphs embedded in $\mathbb{R}^m$, i.e. we will try to determine which sofic ensembles can be limit points of sequences of random graphs embedded in $\mathbb{R}^m$. For this we need to describe the *entropy* of a local ensemble, as well as describe the number of spanning trees of a graph in a local way.

Remark 8.1.5. Local ensembles may also be defined using finite graphs with a designated vertex, instead of a designated edge. It is not difficult to go from one description to another, especially when working with regular graphs.

8.1.3 Entropy of a local ensemble

A probability distribution $\mu_R \in \mathcal{P}(\mathcal{L}_R)$ is called *sofic* if for each $\varepsilon > 0$ there is a d-regular graph G with diameter at least $2R$ such that $\rho_R(G) \overset{\varepsilon}{\sim} \mu_R$. For such a probability distribution we try to define an *entropy* which describes *how many* graphs G on n vertices approximate μ_R locally. The idea is similar to the entropy described in [6], but there only (not necessarily regular) trees are considered, whereas we consider ensembles which may contain cycles.

Definition 8.1.6. Let $\mu_R \in \mathcal{P}(\mathcal{L}_R)$ be a sofic probability distribution. We say that μ_R *has entropy* $h(\mu_R)$ if, for each $\varepsilon > 0$, the number of d-regular graphs on n vertices with $\rho_R(G) \overset{\varepsilon}{\sim} \mu_R$ is $\exp((h(\mu_R) + o(1))n\log(n))$.

It remains unclear whether the entropy always exists. However, we can prove this under an assumption regarding combinatorial cost. Recall the definitions about cycle spaces $Z_R(G)$ and the combinatorial cost $c(G)$ from Definitions 3.1.1 and 3.1.3. In [13, Proposition 2.3] it is shown that, for a d-regular graph sequence G_m, the combinatorial cost can be bounded from below by the limit of $\frac{d}{2} - \frac{\dim Z_R(G_m)}{|V(G_m)|}$. Moreover they ask the following question:

Question 8.1.7 (from [13], Question 2.5, adapted to regular graphs)**.** *Is it true that for all d-regular graph sequences G_m we have*

$$c((G_m)) = \inf_R \liminf_m \frac{d}{2} - \frac{\dim Z_R(G_m)}{|V(G_m)|}?$$

Under the assumption that the answer to this Question is positive, we can prove the existence of entropy.

Theorem 8.A. Suppose the answer to Question 8.1.7 is positive. Let $\mu_R \in \mathcal{P}(\mathcal{L}_R)$ be a sofic probability distribution. There exists $1 \leq h(\mu_R) \leq \frac{d}{2}$, called the *entropy* of μ_R, such that for each $\varepsilon > 0$, the number of d-regular graphs on n vertices with $\rho_R(G) \overset{\varepsilon}{\sim} \mu_R$ is $\exp((h(\mu_R)+o(1))n\log(n))$.

The implied constant in the $o(1)$ notation depends on ε and n and goes to zero as $n \to \infty$ and then $\varepsilon \to 0$. For a sofic ensemble μ, note that $h(\mu_{R'}) \leq h(\mu_R)$ for all $R' > R$; after all, any graph G for which $\rho_{R'}(G) \overset{\varepsilon}{\sim} \mu_{R'}$ also satisfies $\rho_R(G) \overset{\varepsilon}{\sim} \mu_R$. The entropy of μ is then defined as $h(\mu) = \inf_R h(\mu_R)$. We do not have a good way to compute $h(\mu_R)$, but for $h(\mu)$ we have a good upper bound. For a finite connected graph A, and an ordering $\preceq$ on A, the *minimum spanning tree* $\mathrm{MST}(A, \preceq)$ on A is the tree generated by Kruskal's algorithm (see [29]): starting from the empty graph on the vertex set of A, at each step the least vertex is added that does not create a cycle, ending up with a spanning tree for A. Taking $\preceq$ uniformly at random from all possible orderings we get the *random minimum spanning tree* $\mathrm{RMST}(A)$ (see e.g. [35] for more information).

Theorem 8.B. Let μ be a sofic ensemble with an entropy. Let $R > 0$ and let (A, e_*) be a random element of $\mathcal{L}_R$ with probability distribution μ_R. Let $\mathrm{RMST}(A)$ be a random minimum spanning tree on A. Then

$$h(\mu) \leq \frac{d}{2}\mathbb{P}_{\mu_R}(e_* \in \mathrm{RMST}(A)).$$

We will see in several examples that equality holds when we take the infimum over all R. In fact, we have no counterexamples to this. Furthermore, in all examples we know, the right-hand side also equals the average resistance distance of an edge (which can be defined in a similar way, using a uniformly random spanning tree instead of a random minimal spanning tree).

Examples 8.1.8. (i) The free local ensemble μ_{free} has entropy equal to $\frac{d}{2}$. Indeed, almost all d-regular graphs on n vertices have very few small cycles, and thus the local ensemble approximates the free one. For all R we also have $\frac{d}{2}\mathbb{P}_{\mu_{\text{free},R}}(e_* \in \mathrm{RMST}(A)) = \frac{d}{2}$

(ii) Consider a linear ensemble λ_H. From Proposition 8.3.1 we see that $h(\lambda_H) \geq 1$. Construct G as in Example 8.1.4(ii). Let R be larger then the diameter of H. Pick e uniformly as one of the edges of H_0 or the edge between H_0 and H_1. Then $(B_R(e), e)$ is a random element of $\mathcal{L}_R$ chosen according to λ_H. If e is the edge between H_0 and H_1 then it is certainly in any spanning tree of $B_R(e)$. Any spanning tree of $B_R(e)$ is in particular a spanning tree of H_0, and then it contains $|V(H_0)| - 1$ of its edges. Thus the probability that e is in this spanning tree is $\frac{|V(H_0)|}{|E(H_0)+1|} = \frac{2}{d}$. We find that $\mathbb{P}_{\lambda_{H,R}}(e_* \in \mathrm{RMST}(A)) = \frac{2}{d}$. So $h(\lambda_H) = \frac{d}{2}\mathbb{P}_{\lambda_{H,R}}(e_* \in \mathrm{RMST}(A)) = 1$.

8.1.4 Spanning trees

Let G be a connected d-regular graph. For every vertex x and positive integer k, denote by $p_k(x)$ the probability that a random walk starting at x is at x after k steps. An important ingredient in describing the local behaviour of the random graphs embedded in $\mathbb{R}^m$ is the following Proposition by Lyons (see [34]).

Proposition 8.1.9 ([34], Proposition 3.1)**.** *Let G be a finite connected d-regular graph with n vertices and number of spanning trees $T(G)$. Then*

$$\log(T(G)) = -\log(|E(G)|) + n\log(d) - \sum_{k=1}^{\infty} \frac{1}{k}\left(\sum_{x \in V(G)} p_k(x) - 1\right).$$

□

For a connected d-regular graph G on n vertices and $x \in V(G)$ we use the notation

$$\tau(x) = \log(d) - \sum_{k=1}^{\infty} \frac{1}{k} \left(p_k(x) - \frac{1}{n} \right).$$

For an infinite connected d-regular graph G and $x \in V(G)$ we also define

$$\tau(x) = \log(d) - \sum_{k=1}^{\infty} \frac{1}{k} p_k(x).$$

Let $(A, e_*) \in \mathcal{L}_R$. Consider a random walk starting at a random endpoint of e_*, which has a probability of $1 - \frac{\deg(y)}{d}$ to stop each time it visits a vertex y. Denote by $p_k(A, e_*)$ the probability that this random walk goes on for at least k steps and is at its starting vertex after k steps. For $\mu_R \in \mathcal{P}(\mathcal{L}_R)$ we define

$$\tau(\mu_R) = \mathbb{E}_{\mu_R} \left(\log(d) - \sum_{k=1}^{\infty} \frac{1}{k} p_k(A, e_*) \right).$$

For a local ensemble μ, note that $\tau(\mu_{R'}) \geq \tau(\mu_R)$ for $R' > R$. We let $\tau(\mu) = \sup_R \tau(\mu_R)$. Then we can prove that if G_m converges locally to μ, then $\lim \frac{\log T(G_m)}{|V(G_m)|} = \tau(\mu)$ (see Proposition 8.5.1). Also, τ takes finite values (see [34, Corollary 3.9]).

Examples 8.1.10. (i) For the free ensemble μ_{free} we have

$$\tau(\mu_{\text{free}}) = (d-1)\log(d-1) - (\frac{d}{2} - 1)\log(d(d-2)),$$

see [34, Example 3.16].

(ii) For a linear ensemble λ_H it is easier to compute $\tau(\lambda_H)$ by computing the number of spanning trees of the graphs G_m as defined in Example 8.1.4, and using Proposition 8.5.1. We get $T(G_m) \sim T(H)^m$ and thus $\tau(\lambda_H) = \frac{\log T(H)}{|V(H)|}$.

8.1.5 Local description of random graphs embedded in $\mathbb{R}^m$

Let (G_m) be a sequence of d-regular random graphs embedded in $\mathbb{R}^m$ with $n(m)$ vertices. We will study the aymptotic local behaviour of the sequence (G_m), under the assumption that every sofic ensemble has an entropy. It turns out that it depends on the growth of n. Assume that $c = \lim_m \frac{\log(n)}{m}$ exists.

Definition 8.1.11. A sofic ensemble μ is called *c-optimal* if it attains the infimum

$$\tau(\mu) - c \cdot h(\mu) = \inf\{\tau(\mu') - c \cdot h(\mu') \mid \mu' \text{ sofic ensemble}\}.$$

By compactness, a c-optimal local ensemble exists for every c.

Theorem 8.C. Suppose that every sofic ensemble has an entropy. Let (G_m) be a sequence of random graphs with n vertices embedded in $\mathbb{R}^m$, where $n = n(m)$ satisfies $\lim_{m\to\infty} \log(n)/m = c$. Then almost surely every local limit point of the sequence (G_m) is c-optimal.

For large c the only c-optimal ensemble is the free one (see Proposition 8.5.2). Thus if n increases fast enough in m, we see that the sequence of graphs G_m will almost surely have large girth. For $c = 0$ the entropy is not relevant and the 0-local ensemble is $\mu_{\min}$, the sofic ensemble which minimises $\tau(\mu_{\min})$.

We did not find out what the c-optimal sofic ensembles are for all d and all c. However we put forth some conjectures.

Conjectures 8.1.12. *(i) For $d = 3$, let H be the simple graph on 4 vertices with 5 edges. The minimal sofic ensemble is λ_H.*

(ii) In general the minimal sofic ensemble is always a linear one.

(iii) There is a critical value $c_{\text{crit}} > 0$, such that for all $c < c_{\text{crit}}$ the only c-optimal ensemble is $\mu_{\min}$ and for $c > c_{\text{crit}}$ the only c-optimal ensemble is c_{free}.

If the second conjecture is true, then to find the minimal ensemble we need to minimise $\frac{T(H)}{|V(H)|}$ over d-regular graphs from which a vertex has been removed. See [1] for more information.

If the last conjecture is true then we can almost completely describes the local behaviour of the graphs G_m (except when $c = c_{\text{crit}}$). The idea is that, if the free ensemble is not the c-optimal one, it is beneficial to add edges to the local graphs, because they reduce the number of spanning trees more then they reduce the entropy. But then each new edge we add reduces the number of spanning trees by more then it reduces the entropy, thus we want to keep adding edges until we have the minimal number of spanning trees. It is however not easy to make these statements precise, let alone prove them. It might be possible to prove the last conjecture using Theorem 8.B, in such a way that it is not even necessary to prove the existence of entropy in general, thus avoiding the need to answer Question 8.1.7.

8.2 Embeddings in $\mathbb{R}^m$

Now we will consider the manifolds $\mathbb{R}^m$ with base-point zero. For these manifolds, we can compute $\varphi_0(G, \mathbb{R}^m)$ in terms of the number $T(G)$ of spanning trees of G.

Proposition 8.2.1. *Let G be a graph on X. Then*

$$\varphi(G, \mathbb{R}) \sim T(G)^{-\frac{1}{2}}.$$

It follows that

$$\varphi(G, \mathbb{R}^m) \sim T(G)^{-\frac{m}{2}}.$$

Proof. Pick a direction for each edge of G. Let T be the set of edges of a spanning tree of G and let $B = E \setminus T$. Then we consider the following pairing between B and T: for $b \in B$, consider the unique path in T from the source of b to the range of b. Then put $\langle b, t\rangle = 1$ if the path goes through t in the positive direction and -1 if the path goes through t in the negative direction. Otherwise, put $\langle b, t\rangle = 0$. Let C be the $(n-1) \times (dn/2 - n + 1)$-matrix corresponding to this pairing.

There is a one-to-one correspondence between functions $\gamma\colon X \to \mathbb{R}$ with $\gamma(1) = 0$ and functions with the same name $\gamma\colon \mathbb{C}[E] \to \mathbb{R}$ with $\gamma(c) = 0$ for every cycle c. This correspondence is given by $\gamma(e) = \gamma(y) - \gamma(x)$ for every edge $e = (x, y) \in E$. Next, every $\gamma\colon \mathbb{C}[E] \to \mathbb{R}$ that is zero on all cycles is determined by its values on T. For $b \in B$ we have $\gamma(b) = \sum_{t \in T}\langle b, t\rangle\gamma(t)$.

So

$$\varphi_0(G,\gamma) = \exp\left(-\sum_{t\in T}\gamma(t)^2 + \sum_{b\in B}\left(\sum_{t\in T}\langle b,t\rangle\gamma(t)\right)^2\right) = \exp\left(-\sum_{t,t'\in T}(I+CC^T)_{tt'}\gamma(t)\gamma(t')\right).$$

Now let Q be an upper triangular $(n-1)\times(n-1)$-matrix with ones on the diagonal such that $Q^T(1+CC^T)Q = D$ is a diagonal matrix. Then

$$\varphi_0(G) = \int_\gamma \exp\left(-\sum_{t,t'\in T}(Q^{-T}DQ^{-1})_{tt'}\gamma(t)\gamma(t')\right).$$

By a linear change of variables, this is equal to

$$\int_\gamma \exp\left(-\sum_t D_{tt}\gamma(t)^2\right) = \sqrt{\frac{\pi^{n-1}}{\det(D)}} = \sqrt{\frac{\pi^{n-1}}{\det(1+CC^T)}} = \sqrt{\frac{\pi^{n-1}}{\det(1+C^TC)}}.$$

By [3, Theorem 5.7] we know that $\det(1+C^TC)$ equals the number of spanning trees in G, so it follows that $\varphi(G,\mathbb{R}) \sim T(G)^{-\frac{1}{2}}$.

Now we consider the case that $M = \mathbb{R}^m$ for some m, with basepoint 0. Any function $\gamma\colon X \to M$ can then be written $\gamma = (\gamma_1,\ldots,\gamma_n)$ with each $\gamma_i\colon X \to \mathbb{R}$. We have

$$\varphi_0(G,\gamma) = \prod_{(x,y)\in E}\exp(-m\|\gamma(x),\gamma(y)\|^2) = \prod_{i=1}^m\prod_{(x,y)\in E}\exp(-m|\gamma_i(x)-\gamma_i(y)|^2).$$

So we see that for any graph G on X, we have

$$\begin{aligned}
\varphi_0(G,\mathbb{R}^m) &= \left(\int_{\gamma\colon X\to\mathbb{R},\gamma(1)=0}\prod_{(x,y)\in E}\exp(-m|\gamma(x)-\gamma(y)|^2)\right)^m\\
&= \left(m^{-n/2}\int_{\gamma\colon X\to\mathbb{R},\gamma(1)=0}\prod_{(x,y)\in E}\exp(-|\gamma(x)-\gamma(y)|^2)\right)^m\\
&= m^{-n/2}\varphi_0(G,\mathbb{R})^m,
\end{aligned}$$

using the substitution $\gamma \to m^{-1/2}\gamma$. So $\varphi(G,\mathbb{R}^m) \sim T(G)^{-\frac{m}{2}}$. □

Remark 8.2.2. By Kirchoff's matrix tree theorem, this is also proportional to the product of the positive eigenvalues of the Laplacian operator Δ_G.

The asymptotic distribution of the number of spanning trees in a uniformly random regular graph is computed in [18]. However, since the random graphs embedded in $\mathbb{R}^m$ according to our model are heavily weighted towards having fewer spanning trees, this does not give us enough information.

Now we consider how to construct a random function $\gamma\colon X \to \mathbb{R}$ according to φ, when the graph $G = (X,E)$ is already given. Let $\nabla\colon \mathbb{C}[X] \to \mathbb{C}[E]$ denote the function $\nabla(f)((x,y)) =$

$f(y) - f(x)$ for $f \in \mathbb{C}[X]$ and $(x, y) \in E$. Let $\nabla\colon \mathbb{C}[E] \to \mathbb{C}[X]$ also denote the function $\nabla(g)(x) = \sum_{(y,x)\in E} g((y, x))$ for $g \in \mathbb{C}[E]$ and $x \in X$. These are adjoint in the sense that $\langle \nabla f, g\rangle = \langle f, \nabla g\rangle$ using the standard inner products on $\mathbb{C}[X]$ and $\mathbb{C}[E]$. We also have $\Delta_G = \nabla^2\colon \mathbb{C}[X] \to \mathbb{C}[X]$. Since G is assumed to be a connected graph, we know that the kernel of Δ_G consists of the constant functions on X. We denote by Δ_G^{-1} the pseudo-inverse of Δ_G. Define $p\colon \mathbb{C}[E] \to \mathbb{C}[E]$ by $p = \nabla\Delta_G^{-1}\nabla$. It is then easy to check that p is the projection on $C(G)^\perp$, where $C(G) = \ker(\nabla) \subseteq \mathbb{C}[E]$ denotes the cycle space. Define the bilinear form V on $\mathbb{C}[E]$ by $V(g, h) = \langle pg, h\rangle$.

Proposition 8.2.3. *The following give the same probability distribution on the space of functions $\gamma\colon X \to \mathbb{R}$ with $\gamma(1) = 0$.*

(i) Take each $\gamma\colon X \to \mathbb{R}$ with $\gamma(1) = 0$ with probability proportional to

$$\varphi_0(G, \gamma) = \prod_{(x,y)\in E} \exp(-m|\gamma(x) - \gamma(y)|^2).$$

(ii) Generate a random $\gamma\colon E \to \mathbb{R}$ by drawing each $\gamma(e)$ independently from the normal distribution with variation $\frac{1}{2m}$. Condition on the condition $\gamma(c) = 0$ for each cycle. Then define $\gamma\colon X \to \mathbb{R}$ by picking for each $x \in X$ a path $e_1, \ldots, e_k$ from 1 to x and putting $\gamma(x) = \gamma(e_1) + \ldots + \gamma(e_k)$.

(iii) Generate a random function $\gamma\colon E \to \mathbb{R}$ from the joint normal distribution using the variance matrix $\frac{1}{2m}V$. Then define $\gamma\colon X \to \mathbb{R}$ by picking for each $x \in X$ a path $e_1, \ldots, e_k$ from 1 to x and putting $\gamma(x) = \gamma(e_1) + \ldots + \gamma(e_k)$.

Proof. Consider the second way to generate γ. The intensity of any function $\gamma\colon E \to \mathbb{R}$ that is zero on cycles is equal to $\prod_{e\in E} \exp(-m|\gamma(e)|^2)$. So we get the same distribution as in (i).

Now consider the third way to generate γ. We have $\mathbb{C}[E] = C(G) \oplus C(G)^\perp$, and for $c \in C(G)$ and $g \in \mathbb{C}[E]$ we have $V(c, g) = 0$, while for $g, h \in C(G)^\perp$ we have $V(g, h) = \langle g, h\rangle$. So any γ that we generate with the joint normal distribution is zero on all cycles, and for any $\gamma\colon X \to \mathbb{R}$ that is zero on the cycles, the intensity is proportional to $\exp(-mV^{-1}(\gamma, \gamma)) = \exp(-m\langle\gamma, \gamma\rangle) = \prod_{e\in E} \exp(-m|\gamma(e)|^2)$. $\square$

From this we can find the distribution of the distance between two fixed points in the graph.

Proposition 8.2.4. *Let G be a graph and let $\gamma\colon X \to \mathbb{R}^m$ be a random embedding. Let $x, y \in X$. Then $d(\gamma(x), \gamma(y))$ is distributed as $\sqrt{\frac{1}{2m}\langle\Delta_G^{-1}(x - y), x - y\rangle}\chi_m$ where χ_m is chi-distributed with m degrees of freedom.*

Proof. Consider the first coordinate γ_1. It is distributed as in Proposition 8.2.3. Let $e_1, \ldots, e_k$ be a path from x to y in the graph. Then we have $\gamma_1(y) - \gamma_1(x) = \gamma_1(e_1 + \ldots + e_k)$. So $\gamma_1(y) - \gamma_1(x)$ is normally distributed with variance

$$\frac{1}{2m}V(e_1+\ldots+e_k, e_1+\ldots+e_k) = \frac{1}{2m}\langle\nabla\Delta_G^{-1}\nabla(e_1+\ldots+e_k), e_1+\ldots+e_k\rangle = \frac{1}{2m}\langle\Delta_G^{-1}(y-x), y-x\rangle.$$

The same analysis holds for the other coordinates, and the coordinates are independent. So $d(\gamma(x) - \gamma(y)) = \left(\sum_{i=1}^m |\gamma_i(x) - \gamma_i(y)|^2\right)^{\frac{1}{2}}$ is distributed as $\sqrt{\frac{1}{2m}\langle\Delta_G^{-1}(x - y), x - y\rangle}\chi_m$. $\square$

8.3 Proof of Theorem 8.A

We will prove that $h(\mu_R)$ can be computed from the maximal dimension of the small cycle spaces of graphs that locally approximate $h(\mu_R)$, assuming a positive answer to Question 8.1.7. First we show two propositions, to give a lower bound and an upper bound.

Proposition 8.3.1. *Let H be a graph on k vertices satisfying $\dim Z_{2d^R}(H) = \alpha k$. For large n, the number of graphs G on a set of n vertices satisfying $\rho_R(G) = \rho_R(H)$ is at least $\exp((\frac{d}{2} - \alpha)n \log n + O(n))$.*

Proof. Let $R' = 2d^R$. For a subset $E' \subseteq E(H)$ let $H \setminus E'$ denote the graph on the same vertex set as H and with edge set $E(H) \setminus E'$. Let $a = \alpha k$ and choose recursively $e_1, \dots, e_a$ such that e_{j+1} is an edge in a cycle in $Z_{R'}(H) \cap Z(H \setminus \{e_1, \dots, e_j\})$. Then $\dim Z_{R'}(H) \cap Z(H \setminus \{e_1, \dots, e_j\}) = a - j$. Let $e_{a+1}, \dots, e_{|E(H)|}$ be the other edges of H.

We can assume that n is a multiple of k. Let X be the vertex set of H. We will create graphs G on the vertex set $X \times \{1, \dots, n/k\}$. For each edge (x, y) of H we will choose a permutation $\sigma_{x,y}$ of $\{1, \dots, n/k\}$, satisfying $\sigma_{x,y} = \sigma_{y,x}^{-1}$. Then in G we will connect (x, i) to $(y, \sigma_{x,y} i)$ for all $i \in \{1, \dots, n/k\}$.

We will determine the σ_{e_j} backwards. For $j > a$ we can choose them in any way. Then for $j \leq a$, there is a cycle in $Z_{R'}(H) \cap Z(H \setminus \{e_1, \dots, e_{j-1}\})$ containing e_j. Let σ_{e_j} be the product of the permutations corresponding to the other edges in this cycle. Then for any cycle $e_{j_1}, e_{j_2}, \dots, e_{j_s}$ in $Z_R(H)$ we then get $\sigma_{e_{j_1}} \circ \sigma_{e_{j_2}} \circ \dots \circ \sigma_{e_{j_s}} = 1$. The σ_{e_j} does not depend on the specific cycle we choose.

Now we have a graph morphism $G \to H$ sending the vertex (x, i) to x. Moreover all cycles in H that are visible in the R-neighbourhood around an edge have length at most R', so they lift to a cycle in G. So the R-neighbourhood of an edge $((x, i), (y, \sigma_{x,y} i))$ in G is isomorphic to the R-neighbourhood of the edge (x, y) in H. So $\rho_R(G) = \rho_R(H)$. We could choose the σ_{e_j} for $j > a$ in any way, and each different way will give a different graph on $X \times \{1, \dots, n/k\}$. Thus there are at least $\big((n/k)!\big)^{|E(H)|-a} = \exp((\frac{d}{2} - \alpha)k \log((n/k)!))$ possible graphs G. Using Stirling's approximation this is $\exp((\frac{d}{2} - \alpha)n \log(n) + O(n))$. □

Proposition 8.3.2. *Suppose the answer to Question 8.1.7 is positive. Let $\alpha > 0$ and let R be a positive integer. There are at most $\exp((\frac{d}{2} - \alpha + o(1))n \log n)$ d-regular graphs G on $X = \{1, \dots, n\}$ satisfying $\dim Z_R(G) \geq \alpha n$.*

Proof. Let $\varepsilon > 0$. Consider the sequence containing all possible d-regular graphs G satisfying $\dim Z_R(G) \geq \alpha |V(G)|$ up to isomorphism. Our assumption entails that the combinatorial cost of this sequence is $\frac{d}{2} - \alpha$. So there is a graph sequence which induces the same coarse structure, for which every graph has at most $\frac{d}{2} - \alpha + \varepsilon$ as many edges as vertices. Then there is a constant L such that for every d-regular graph G on $X = \{1, \dots, n\}$ with $\dim Z_R(G) \geq \alpha n$ there is a graph H on the same set of vertices, such that for all $x, y \in V(G)$ we have $\frac{1}{L} d_G(x, y) \leq d_H(x, y) \leq L d_G(x, y)$, and $|E(H)| \leq (\frac{d}{2} - \alpha + \varepsilon)n$.

There are $\exp((\frac{d}{2} - \alpha + \varepsilon)n \log n + O(n))$ graphs H on X satisfying $|E(H)| \leq (\frac{d}{2} - \alpha + \varepsilon)n$. We claim that for any such H there are only exponentially many d-regular graphs G satisfying $\frac{1}{L} d_G(x, y) \leq d_H(x, y) \leq L d_G(x, y)$ for all $x, y \in X$. First of all note that if at least one such G exists, then every vertex has at most d^L neighbours in H. Now if (x, y) is an edge in G, then the distance in H between them is at most L, so if x is fixed then there are at most d^{L^2} possibilities for y. Since every vertex will have d edges the total number of possible graphs G is at most $d^{L^2 dn}$ which is only exponential in n.

Thus the number of d-regular graphs on X satisfying $\dim Z_R(G) \geq \alpha n$ is at most $\exp((\frac{d}{2} - \alpha + \varepsilon)n \log n + O(n))$. Since this holds for all $\varepsilon > 0$ it is also at most $\exp((\frac{d}{2} - \alpha + o(1))n \log n)$. $\square$

Now we can prove Theorem 8.A.

Proof of Theorem 8.A. Let α_ε be the infimum of $\frac{\dim Z_{2d^R}(H)}{|V(H)|}$, where H ranges over all graphs with $\rho_R(H) \overset{\varepsilon}{\sim} \mu_R$. By Proposition 8.3.1, the number of d-regular graphs G on n vertices satisfying $\rho_R(G) \overset{\varepsilon}{\sim} \mu_R$ is at least $\exp((\frac{d}{2} - \alpha_\varepsilon + o(1))n \log n)$. For all such graphs we have $\dim Z_{2d^R}(G) \geq \alpha_\varepsilon |V(G)|$, so by Proposition 8.3.2 they also number at most $\exp((\frac{d}{2} - \alpha_\varepsilon + o(1))n \log n)$. So their number is $\exp((\frac{d}{2} - \alpha_\varepsilon + o(1))n \log n)$, and letting $\varepsilon \to 0$ we find $h(\mu_R) = \frac{d}{2} - \sup_\varepsilon \alpha_\varepsilon$. $\square$

8.4 Proof of Theorem 8.B

Now we give the proof of Theorem 8.B, which we restate for convenience. The idea of the proof is to order the edges of G randomly and to draw them one by one. The edges that are contained in a small cycle with preceding edges can then only be drawn in one way. Of course the problem with this idea is that the edges of G can not be ordered if G is not already defined, so to get around this we order them based on some neighbourhood. To make this possible we introduce decorated neighbourhoods to make sure that edges that are close together will have a different neighbourhood.

Theorem 8.B. Let μ be a sofic ensemble with an entropy. Let $R > 0$ and let (A, e_*) be a random element of $\mathcal{L}_R$ with probability distribution μ_R. Let $\mathrm{RMST}(A)$ be a random minimum spanning tree on A. Then

$$h(\mu) \leq \frac{d}{2} \mathbb{P}_{\mu_R}(e_* \in \mathrm{RMST}(A)).$$

Proof. Let $k = d^{2R+1}$. The set of *decorated local graphs* $\mathcal{DL}_{2R}$ consists of all elements $(A, e_*) \in \mathcal{L}_{2R}$ together with an injective function from A to $\{1, \ldots, k\}$. Similarly $\mathcal{DL}_R$ consists of elements (A, e_*) with an injective function from A to $\{1, \ldots, k\}$. For a finite graph G, a function χ from the edge set E to $\{1, \ldots, k\}$ is called *suitable* if no two edges within distance $2R$ have the same colour. Every d-regular graph has a suitable colouring, since for every edge there are less then k other edges within a distance of $2R$. For a graph G with suitable colouring χ we let $\rho'_{2R}(G, \chi) \in \mathcal{P}(\mathcal{DL}_{2R})$ be the corresponding *decorated local ensemble*: it is the probability measure on $\mathcal{DL}_{2R}$ we get from choosing a random edge in G and considering its $2R$-neighbourhood. We also have a projection $p \colon \mathcal{P}(\mathcal{DL}_{2R}) \to \mathcal{P}(\mathcal{L}_{2R})$ that simply forgets about the colouring. We have $p(\rho'_{2R}(G, \chi)) = \rho_{2R}(G)$.

We will estimate the number of d-regular graphs G on n vertices for which $\rho_{2R}(G) \overset{\varepsilon}{\sim} \mu_{2R}$. Since the preimage $p^{-1}(\mu_{2R}) \subseteq \mathcal{P}(\mathcal{DL}_{2R})$ can be covered by finitely many balls or radius ε, and every d-regular graph has a suitable colouring, it is enough to fix $\mu'_{2R} \in p^{-1}(\mu_{2R})$ and bound the number of graphs G with a suitable colouring χ for which $\rho_{2R}(G, \chi) \overset{2\varepsilon}{\sim} \mu'_{2R}$.

Let $\preceq$ be a uniformly random total ordering of the finite set $\mathcal{DL}_R$. Let $(A, e_*, \chi) \in \mathcal{DL}_{2R}$. Then for all $e \in B_R(e_*)$, the R-ball around e is an element of $\mathcal{DL}_R$. They are all different since they have a different colour. Thus the edges in an R-neighbourhood of e_* are ordered according to $\preceq$.

This is a uniformly random ordering. Let $\mathrm{MST}(\preceq)$ be the corresponding minimum spanning tree of $B_R(e_*)$. Then for fixed (A, e_*, χ) we have

$$\mathbb{P}[e_* \in \mathrm{MST}(\preceq)] = \mathbb{P}[e_* \in \mathrm{RMST}(B_R(e_*))].$$

The left-hand side depends on (A, e_*, χ) but the right-hand side only depends on $(B_R(e_*), e_*) \in \mathcal{L}_R$. Taking a random element of $\mathcal{DL}_{2R}$ according to μ'_{2R} we get

$$\mathbb{P}_{\mu'_{2R}}[e_* \in \mathrm{MST}(\preceq)] = \mathbb{P}_{\mu_R}(e_* \in \mathrm{RMST}(A)).$$

Therefore we can fix an ordering $\preceq_0$ of $\mathcal{DL}_R$ for which

$$\mathbb{P}_{\mu'_{2R}}[e_* \in \mathrm{MST}(\preceq_0)] \leq \mathbb{P}_{\mu_R}(e_* \in \mathrm{RMST}(A)).$$

Let $F \subseteq \mathcal{DL}_{2R}$ be the set of (A, e_*, χ) for which $e_* \in \mathrm{MST}(\preceq_0)$.

Now we will estimate the number of graphs G on $X = \{1, \dots, n\}$ with a suitable colouring χ for which $\rho'_{2R}(G, \chi) \overset{2\varepsilon}{\sim} \mu'_{2R}$. We can assume that the number of edges e for which $(B_{2R}, e, \chi) \in F$ is f, where $f \leq (\mathbb{P}_{\mu_R}(e_* \in \mathrm{RMST}(A)) + 2\varepsilon)\frac{dn}{2}$. For each $x \in X$ we can decide the decorated $2R$-neighbourhoods of the edges adjacent to x. The number of ways to do this grows only exponentially in n. Then we can draw all the edges whose decorated $2R$-neighbourhoods are in F. There are at most $\exp(f \log(n))$ ways to do that. Once we have done that, there can be at most one way to construct G: to see this we construct the edges in the order $\preceq_0$ based on their decorated R-neighbourhood. For each vertex x we already decided what the $2R$-neighbourhoods of all of its edges look like. When an edge gets added whose decorated $2R$-neighbourhood is not in F, then it is contained in a cycle contained in its R-neighbourhood with other edges which were already decided before it. Thus in this case this new edge is also already decided.

This shows that the number of graphs G with a suitable colouring χ for which $\rho'_{2R}(G, \chi) \overset{2\varepsilon}{\sim} \mu'_{2R}$ is at most $\exp((\mathbb{P}_{\mu_R}(e_* \in \mathrm{RMST}) + 2\varepsilon)\frac{dn}{2}\log(n) + O(n))$. In particular, the number of graphs G for which $\rho_{2R}(G) \overset{\varepsilon}{\sim} \mu_{2R}$ is also at most $\exp((\mathbb{P}_{\mu_R}(e_* \in \mathrm{RMST}) + 2\varepsilon)\frac{dn}{2}\log(n) + O(n))$. Therefore

$$h(\mu) \leq h(\mu_{2R}) \leq \frac{d}{2}\mathbb{P}_{\mu_R}(e_* \in \mathrm{RMST}(A)).$$

□

8.5 Local description of random graphs embedded in $\mathbb{R}^m$

In this section we will prove Theorem 8.C. We will also determine the local behaviour of random graphs G_m with n vertices embedded in $\mathbb{R}^m$ when $c = \lim \frac{\log n}{m}$ is large.

First we need this proposition.

Proposition 8.5.1. *Let G_m be a sequence locally converging to a sofic ensemble μ. Then*

$$\lim_{m\to\infty} \frac{\log T(G_m)}{|V(G_m)|} = \tau(\mu).$$

Proof. This can be proved in essentially the same way as [34, Theorem 3.3]. □

Now we can prove the Theorem.

Theorem 8.C. Suppose that every sofic ensemble has an entropy. Let (G_m) be a sequence of random graphs with n vertices embedded in $\mathbb{R}^m$, where $n = n(m)$ satisfies $\lim_{m\to\infty} \log(n)/m = c$. Then almost surely every local limit point of the sequence (G_m) is c-optimal.

Proof. Let μ_0 be a c-optimal sofic ensemble. Let $\varepsilon > 0$. From Proposition 8.5.1 it follows that there is an integer R such that for any d-regular graph G on n vertices, for n large enough, we have

$$\left| \frac{\log T(G)}{n} - \tau(\rho_R(G)) \right| < \varepsilon.$$

By continuity there is $\varepsilon' > 0$ such that if $\mu, \mu' \in \mathcal{P}(\mathcal{L}_R)$ are ε'-close, then $|\tau(\mu) - \tau(\mu')| < \varepsilon$. Then by compactness there are sofic ensembles $\mu_1, \mu_2, \dots, \mu_N \in \mathcal{P}(\mathcal{L}_R)$ such that for $1 \le k \le N$ we have $\tau(\mu_k) - c \cdot h(\mu_k) \ge \tau(\rho_0) - c \cdot h(\rho_0) + 8\varepsilon$, and for every sofic ensemble $\mu \in \mathcal{P}(\mathcal{L}_R)$ with $\tau(\mu) - c \cdot h(\mu) \ge \tau(\rho_0) - c \cdot h(\rho_0) + 8\varepsilon$ there is a $1 \le k \le N$ such that μ is ε'-close to μ_k.

Let m be large enough that every d-regular graph on n vertices has diameter larger than R. Now we estimate the probability that $\rho_R(G_m)$ is ε'-close to μ_k. The number of graphs G for which $\rho_R(G)$ is ε'-close to μ_k is most $\exp((h(\mu_k) + o(1))n \log(n)) \le \exp(mn(c \cdot h(\mu_k) + \varepsilon))$ if m is large enough. For such a graph G we have

$$T(G)^{-m} = \exp(-m \log T(G)) \le \exp\big(mn(-\tau(\rho_R(G) + \varepsilon)\big) \le \exp\big(mn(-\tau(\mu_k) + 2\varepsilon)\big).$$

Hence

$$\begin{aligned} \sum_{G \text{ with } \rho_R(G) \overset{\varepsilon'}{\sim} \mu_k} T(G)^{-m} &\le \exp\Big(mn\left(c \cdot h(\mu_k) - \tau(\mu_k) + 3\varepsilon\right)\Big) \\ &\le \exp\Big(mn\left(c \cdot h(\mu_0) - \tau(\mu_0) - 5\varepsilon\right)\Big). \end{aligned}$$

Now we compare this with the contribution of graphs G for which $\rho_R(G)$ is ε'-close to $\mu_{0,R}$. The number of such graphs is at least $\exp\big((h(\mu_{0,R}) + o(1))n \log(n)\big) \ge \exp\big(mn(c \cdot h(\mu_0) - \varepsilon)\big)$ for large m. For such a graph G we have

$$T(G)^{-m} \ge \exp\big(mn(\tau(\rho_R(G)) - \varepsilon)\big) \ge \exp\big(mn(\tau(\mu_{0,R}) - 2\varepsilon)\big) \ge \exp(mn(h(\mu_0) - 3\varepsilon)).$$

So

$$\sum_{G \text{ with } \rho_R(G) \overset{\varepsilon'}{\sim} \mu_{0,R}} T(G)^{-m} \ge \exp\Big(mn\left(c \cdot h(\mu_0) - \tau(\mu_0) - 4\varepsilon\right)\Big).$$

Comparing these sums, we find that

$$\mathbb{P}[\rho_R(G_m) \text{ is } \varepsilon'\text{-close to } \mu_k] \le \exp(-\varepsilon mn).$$

Since this decreases quickly as $m \to \infty$, we can conclude that this almost surely does not happen for infinitely many values of m, for any k. So for large values of m we have almost surely that $h(\rho_R(G_m)) - c \cdot h(\rho_R(G_m)) \le h(\rho_0) - c \cdot h(\rho_0) + 8\varepsilon$. Since the same inequality holds almost surely for arbitrarily large R, we get that almost surely every local limit point μ of (G_m) satisfies $\tau(\mu) - c \cdot h(\mu) < \tau(\mu_0) - c \cdot h(\mu_0) + 8\varepsilon$. Since this holds for any $\varepsilon > 0$, we conclude that almost surely all local limit points of (G_m) are c-optimal. ◻

Finally we consider what happens for large c.

Proposition 8.5.2. *For large enough c the only c-optimal ensemble is μ_{free}.*

Proof. For $(A, e_*) \in \mathcal{L}_R$ and x one of the endpoints of e_*, consider the lazy random walk starting at x, and at each step doing nothing with a probability of $\frac{1}{2}$, stopping with a probability of $\frac{d-\deg(y)}{2d}$ when at vertex y and otherwise taking one of the edges at random. Let $q_k(x)$ be the probability that this random walk takes at least k steps and is back at x after k steps. For any ensemble μ we have $\tau(\mu) = \lim_{R\to\infty} \mathbb{E}_{\mu_R} \log(d) - \sum_{k=1}^{\infty} \frac{1}{k} p_k(x)$. By [34, Lemma 3.5], this is also equal to $\lim_{R\to\infty} \mathbb{E}_{\mu_R} \log(\frac{d}{2}) - \sum_{k=1}^{\infty} \frac{1}{k} q_k(x)$.

Now let $g < R$ and suppose that x is not contained in any cycle of length smaller than g. We will estimate the sum $\sum_{k=g}^{\infty} \frac{1}{k} q_k(x)$. For $g \leq k \leq g^4$, if the lazy random walk is to end up at x after k steps, then the last $g-1$ steps do not contain any cycles. Thus $q_k(x)$ is at most the return probability of a lazy random walk in a d-regular tree after $g-1$ steps. This decreases exponentially in g. For $k > g^4$ we use the estimate $q_k(x) \leq \frac{4d}{\sqrt{k+1}}$ from [34, Lemma 3.6]. We get $\sum_{k=g^4+1}^{\infty} \frac{1}{k} q_k(x) \leq \sum_{k=g^4+1}^{\infty} \frac{4d}{k\sqrt{k+1}} \sim g^{-2}$. Combining these estimates we see that there is a constant a, independent of g, such that

$$\sum_{k=g}^{\infty} \frac{1}{k} q_k(x) \leq a g^{-1}.$$

Also, the first $g-1$ terms $\sum_{k=1}^{g} \frac{1}{k} q_k(x)$ are the same in A as in a d-regular tree. Therefore, we see that

$$\tau(\mu) \geq \tau(\mu_{\mathrm{free}}) - \sup_R \mathbb{E}_{\mu_R} \mathbb{E}_{x \text{ vertex of } e_*} [a g^{-1} \mid g \text{ length of smallest cycle containing } x],$$

where this last term is understood to be zero if x is not contained in any cycle.

Now let $c > a$ and let μ be any sofic ensemble that is not free. For any graph A and any edge e that is contained in a cycle of length g, the probability that e is in a random minimum spanning tree of A is at most $1 - g^{-1}$, since it is not in the minimum spanning tree if it has a larger weight than all other edges in this cycle. Therefore we have by Theorem 8.B:

$$\begin{aligned}
h(\mu) &\leq \inf_R \frac{d}{2} \mathbb{P}_{\mu_R}(e_* \in \mathrm{RMST}(A)) \\
&= \inf_R \frac{d}{2} \mathbb{E}_{\mu_R} \mathbb{E}_{x \text{ vertex of } e_*} \mathbb{E}_{e \text{ random edge from } x} \mathbb{P}(e \in \mathrm{RMST}(A)) \\
&\leq \frac{d}{2} \inf_R \mathbb{E}_{\mu_R} \mathbb{E}_{x \text{ vertex of } e_*} [1 - \frac{2}{d} g^{-1} \mid g \text{ length of smallest cycle containing } x].
\end{aligned}$$

Combining this with the formula for $\tau(\mu)$ derived above we find that

$$\tau(\mu) - c \cdot h(\mu) > \tau(\mu_{\mathrm{free}}) - \frac{cd}{2} = \tau(\mu_{\mathrm{free}}) - c \cdot h(\mu_{\mathrm{free}}).$$

So μ is not c-optimal. □

Remark 8.5.3. This proposition does not rely on the answer to Question 8.1.7, since it just bounds the entropy using Theorem 8.B.

8.6 Further remarks and questions

While the subject of random graphs embedded in $\mathbb{R}^m$ seems very interesting, it is difficult to prove results and many questions remain open. In addition to the conjectures in 8.1.12 we put some other open questions in this section.

Entropy

The notion of entropy described in Section 8.1.3 is central to the discussion of the local behaviour of the random graphs embedded in $\mathbb{R}^m$. However, we were not able to prove the existence of entropy in general.

Question 8.6.1. *The existence of entropy for a sofic ensemble μ is only proven under the assumption of a positive answer to Question 8.1.7. Can we prove it unconditionally?*

If the entropy indeed always exists, we can find an upper bound of the entropy using Theorem 8.B. If we can show that equality is always attained it gives a nice way to compute the entropy of a sofic ensemble.

Question 8.6.2. *Does the formula*

$$h(\mu) = \frac{d}{2} \inf_R \mathbb{P}_{\mu_R} \mathbb{P}(e_* \in \mathrm{RMST}(A))$$

hold for every sofic ensemble μ?

Other manifolds

We described only random graphs embedded in $\mathbb{R}^m$. It would be interesting to consider other manifolds, such as the torus $\mathbb{T}^m$.

Question 8.6.3. *Can we describe the local and global behaviour of random graphs embedded in the torus $\mathbb{T}^m$?*

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